THE MOD-2 STEENROD ALGEBRA AND ITS APPLICATIONS IN COMPUTING HOMOTOPY GROUPS OF SPHERES

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ABSTRACT

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In this master's thesis, we discuss the Steenrod squares and their various applications. The Steenrod squares are cohomology operations of type $(\mathbb{Z}/2, n; \mathbb{Z}/2, n+i)$ for non-negative integers i, and they generate a graded algebra under the Adem relations, which is a much more extensive algebraic structure on cohomology than the ring structure given by the cup product. We discuss the construction and properties of the Steenrod squares and the Steenrod algebra they generate, and we present two applications: a partial solution to the Hopf invariant one problem, and the computation of (stable) homotopy groups of spheres.

Contents

Chapter 1

Introduction

1.1 Overview

In algebraic topology, we study topological spaces with tools from abstract algebra, and one of our main objectives is to find algebraic invariants that can help us distinguish and classify topological spaces up to homeomorphism (or more often up to homotopy equivalence). In a typical master's program, students learn about three algebraic invariants.

The first is the fundamental group. Of course, the fundamental group π_1 is merely a special case $n = 1$ of homotopy groups π_n, but it is worth talking about fundamental groups separately because they are usually the very first algebraic invariant students learn. Using the fundamental group, we can easily distinguish topological spaces like $\mathbb{R}P^2$ (real projective plane) and S^1 (circle). In particular, $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ but

$\pi_1(S^1) \cong \mathbb{Z}$. However, the low-dimensional nature of the fundamental group quickly poses problems when the difference between two topological spaces lives in higher dimensions. For example, $\pi_1(S^2) \cong \pi_1(S^3)$ but we know $S^2 \ncong S^3$.

To remedy this low-dimensional restriction, we then have the homology groups. Homology groups can detect (a certain type of) difference in all dimensions, and thus they can easily distinguish S^2 from S^3. In fact, they can easily distinguish S^n from S^m whenever $n \neq m$. Frequently, we are not satisfied by simply distinguishing two spaces apart, but we want to know whether the two spaces are related in a certain way. For example, one may wonder if S^{n-1} is a retract of S^n. We can readily obtain the answer NO using homology groups. Say there is a retraction $S^n \xrightarrow{r} S^{n-1}$, then the identity map on S^{n-1} factors through $S^{n-1} \xrightarrow{i} S^n \xrightarrow{r} S^{n-1}$, where i is the inclusion. In homology, this induces a factorization of the identity map on $H_{n-1}(S^{n-1})$: $H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(S^n) \xrightarrow{r_*} H_{n-1}(S^{n-1})$. However, $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ but $H_{n-1}(S^n) = 0$, thus such factorization cannot exist.

Taking the dual notions, we obtain the cohomology groups. The contravariant nature of cohomology gives it an advantage over homology. For an arbitrary space X, the obvious diagonal map $X \to X \times X$ induces a map in cohomology: $H^*(X \times X) \to H^*(X)$. Using the Künneth theorem, we can naturally define a multiplication on the cohomology groups, namely the cup product, which turns the cohomology groups into a graded ring. This is a richer algebraic structure than homology groups, allowing us to distinguish more spaces apart. For example, $T = S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology, but their cohomology ring structures differ. However, the ring structure is still not rich enough to distinguish spaces like $S\mathbb{C}P^2$ (suspension of $\mathbb{C}P^2$) and $S^3 \vee S^5$, since the cup product is trivial in both spaces.

In this thesis, we will develop an algebraic structure on cohomology that is significantly richer than the ring structure afforded by the cup product. We are going to develop an infinite family of operations on cohomology, and we will see these operations generate an algebra that is subject to a certain set of relations. The cohomology is then equipped with a module structure over this algebra, or in other words, this algebra then acts on cohomology [AM71]. Topological maps need to respect not only the cup product but also the action of this algebra. We will study these operations and the algebra they generate in detail, and we will also see how this extensive algebraic structure can help us compute homotopy groups of spheres.

1.2 Preliminaries

In this section, we state some preliminary concepts and results that will become useful for developing our topic and performing computations in later chapters. These are all well-known concepts and results in algebraic topology and can be found in almost all graduate-level algebraic topology textbooks. Therefore, we shall omit the proofs for these theorems and refer to [Hat02] whenever needed.

Definition 1.2.1. A space X with only one non-trivial homotopy group $\pi_n(X) \cong G$

is called an *Eilenberg-MacLane space* $K(G,n)$.

Definition 1.2.2. A space X is said to be *n-connected* if $\pi_i(X) = 0$ for all $i \leq n$.

Remark 1.2.3. X is path-connected if $n = 0$ and simply connected if $n = 1$.

Definition 1.2.4. Given a path-connected space X and a positive integer n, we can define the *Hurewicz homomorphism* $h\colon \pi_n(X) \to H_n(X)$ in the following way: choose a generator u_n of $H_n(S^n)$, consider the homotopy class $[f] \in \pi_n(X)$ represented by some $f\colon S^n \to X$, then we let $h([f]) = f_*(u_n)$, where $f_*\colon H_n(S^n) \to H_n(X)$ is induced by f.

Theorem 1.2.5 (Eilenberg-Zilber Theorem)**.** *Let X, Y be any two spaces, then there is a natural chain-homotopy equivalence*

$$AW\colon C_\bullet(X \times Y) \rightleftarrows C_\bullet(X) \otimes C_\bullet(Y)\colon EZ.$$

AW is the Alexander-Whitney map, and EZ is the Eilenberg-Zilber map. In particular, $AW \circ EZ = id_{C_\bullet(X)\otimes C_\bullet(Y)}$ and $EZ \circ AW \sim id_{C_\bullet(X\times Y)}$.

Theorem 1.2.6 (Hurewicz Theorem)**.** *Let X be a $(n-1)$-connected space. If $n = 1$, then the Hurewicz homomorphism h induces an isomorphism*

$$\tilde{h}\colon \pi_1(X)/[\pi_1(X), \pi_1(X)] \longrightarrow H_1(X),$$

where we mod out $\pi_1(X)$ by its commutator subgroup. If $n \geq 2$, then h is an isomorphism in dimensions $0 < i \leq n$ and an epimorphism in dimension $i = n+1$.

Theorem 1.2.7 (Universal Coefficient Theorem for Cohomology)**.** *Let G be an abelian group, if a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split exact sequences*

$$0 \longrightarrow Ext(H_{n-1}(C), G) \longrightarrow H^n(C;G) \longrightarrow Hom(H_n(C), G) \longrightarrow 0.$$

Theorem 1.2.8 (Künneth Theorem)**.** *Let R be a ring and let X, Y be CW-complexes. If the cohomology group $H^k(Y;R)$ is a finitely generated free R-module for all k, then the cross product*

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is an isomorphism.

(As to how this map is defined, we refer to the discussion in [Hat02, Chapter 3, Section 1].)

Theorem 1.2.9 (Hopf-Whitney Theorem)**.** *Let K be a complex of dimension n, and let Y be a $(n-1)$-connected space. Then there is a one-to-one correspondence*

$$[K, Y] \longleftrightarrow H^n(K; \pi_n(Y)).$$

1.3 Contents of the Thesis

Chapter 2: We give the general definition of a primary cohomology operation. We also state a classification theorem that connects cohomology operations to Eilenberg-

MacLane spaces.

Chapter 3: We restrict our attention to a specific family of cohomology operations called the Steenrod squares. These operations are defined on mod-2 cohomology, and we construct them through a series of so-called cup-i products. We also state and prove eight important properties of the Steenrod squares, many of which will be frequently used later.

Chapter 4: We apply the Steenrod squares to give a partial solution to the famous Hopf invariant one problem.

Chapter 5: We show how the Steenrod squares, together with the Adem relations, generate a graded $\mathbb{Z}/2$-algebra – the Steenrod algebra $\mathcal{A}$. We study the algebraic structures of $\mathcal{A}$ and its linear dual $\mathcal{A}^*$, and we also study the Hopf algebra structure of $\mathcal{A}$ by deriving its comultiplication. We give a set of indecomposable generators for $\mathcal{A}$ as a $\mathbb{Z}/2$-algebra, and we also give two different bases for $\mathcal{A}$ as a $\mathbb{Z}/2$-module.

Chapter 6: As two examples, we explicitly compute the integral cohomology of $K(\mathbb{Z}, 2)$ and the (low-dimensional part of the) mod-2 cohomology of $K(\mathbb{Z}/2, 2)$ using the Serre spectral sequence. We then state and prove Serre's theorem, which gives us the mod-2 cohomology of $K(\mathbb{Z}/2, q)$ for any $q \geq 1$.

Chapter 7: We compute the 2-components of the first five stable homotopy groups. The computations in this chapter will utilize results from Chapters 2, 3, 5, 6.

Appendix: For reference, we compose the computations of the transgressions in Chapter 7 into compact tabular forms.

Chapter 2

Cohomology Operations

In this chapter, we give a brief introduction to the general notion of cohomology operations. We will not dwell on and go very deep into this general notion, since we are only interested in a particular family of cohomology operations in the rest of this thesis. In this and the following chapters, we follow the definitions and notations for cohomology operations as found in [MT68, Chapter 1].

2.1 Basic Definitions

Definition 2.1.1. A *(primary) cohomology operation* of type $(\pi, n; G, m)$ is a family of functions $\theta_X : H^n(X;\pi) \to H^m(X;G)$, one for each space X, satisfying the condition: for any map $f : X \to Y$, the following diagram commutes.

$$\begin{array}{ccc} H^n(Y;\pi) & \xrightarrow{f^*} & H^n(X;\pi) \\ \theta_Y \big\downarrow & & \big\downarrow \theta_X \\ H^m(Y;G) & \xrightarrow{f^*} & H^m(X;G) \end{array}$$

In the language of category theory, this definition is saying that a cohomology operation of type $(\pi, n; G, m)$ is a natural transformation between the cohomology functors $H^n(-;\pi)$ and $H^m(-;G)$.

Definition 2.1.2. The specific operation θ_X at a space X is called the *component* of θ at X.

Example 2.1.3. The familiar cup-product square $\alpha \mapsto \alpha \smile \alpha$ is a cohomology operation of type $(\pi, n; \pi, 2n)$ for any n and π. Note that this operation is in general not a group homomorphism. In fact, this is only a group homomorphism when $2 = 0$ in π. To see the cup-product square satisfies the naturality condition, recall we have the property $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Notation 2.1.4. We denote the set of all cohomology operations of type $(\pi, n; G, m)$ by $\mathcal{O}(\pi, n; G, m)$.

For the rest of this chapter, we will assume $n \geq 2$. The reason is that when $n = 1$ the Hurewicz homomorphism fails to be an isomorphism at dimension $n - 1 = 0$. For a 0-connected (i.e. path-connected) space X, we have $\pi_0(X) = 0$ but $H_0(X) \cong \mathbb{Z}$. Hence $\pi_0(X) \ncong H_0(X)$. As we will see below, this leads to problems when we try to apply the Universal Coefficient Theorem for Cohomology. Therefore, we will exclude this case from our discussion below.

Suppose X is a $(n-1)$-connected space, then by definition $\pi_i(X) = 0$ for all $i < n$. In particular, the Hurewicz theorem gives $H_{n-1}(X) \cong \pi_{n-1}(X) \cong 0$. Note this only works when $n \geq 2$. This, together with the Universal Coefficient Theorem for Cohomology, shows $H^n(X;\pi) \cong \mathrm{Hom}(H_n(X), \pi)$, because the Ext term is $\mathrm{Ext}(0,\pi) = 0$. Now if we let $\pi = \pi_n(X)$, then $\mathrm{Hom}(H_n(X), \pi) = \mathrm{Hom}(H_n(X), \pi_n(X))$ contains the isomorphism h^{-1}, where h is the Hurewicz homomorphism (in this case, h is an isomorphism). This leads to the following definition.

Definition 2.1.5. Let X be a $(n-1)$-connected space. The *fundamental class* of X is the cohomology class $\iota_n \in H^n(X;\pi_n(X))$ which corresponds to h^{-1} under the above isomorphism $H^n(X;\pi) \cong \mathrm{Hom}(H_n(X), \pi)$.

Remark 2.1.6. $K(\pi, n)$ has a fundamental class $\iota_n \in H^n(K(\pi,n);\pi)$.

2.2 A Classification Theorem

We will end our short introduction to primary cohomology operations by introducing a classification theorem.

Theorem 2.2.1. *There is a one-to-one correspondence*

$$[X, K(\pi, n)] \longleftrightarrow H^n(X;\pi)$$

$$[f] \longleftrightarrow f^*(\iota_n).$$

$[X, K(\pi,n)]$ *denotes* $Mor(X, K(\pi, n))/\sim$*, where we mod out by homotopy equivalence.*

The proof requires obstruction theory, which goes beyond the scope of this thesis, and thus it will be omitted. A detailed proof can be found in [MT68, Chapter 1]. As two immediate corollaries, we have the following.

Corollary 2.2.2. *There is a one-to-one correspondence*

$$[K(\pi, n), K(\pi', n)] \longleftrightarrow Hom(\pi, \pi').$$

Proof sketch. We have $[K(\pi, n), K(\pi', n)] \cong H^n(K(\pi, n); \pi') \cong \mathrm{Hom}(H_n(K(\pi, n)), \pi')$ $\cong \mathrm{Hom}(\pi_n(K(\pi, n)), \pi') = \mathrm{Hom}(\pi, \pi')$. The first isomorphism follows from Theorem 2.2.1, the second isomorphism follows from the Universal Coefficient Theorem for Cohomology, and the third isomorphism follows from the Hurewicz theorem. □

Corollary 2.2.3. *The homotopy type of $K(\pi, n)$ is determined by π and n. Moreover, the identity map of π determines (up to homotopy) a canonical homotopy equivalence between any two models of $K(\pi, n)$.*

Proof. This follows directly from the above corollary and the Whitehead theorem. □

Remark 2.2.4. The above result justifies a simplification of notation, namely we can write $H^m(\pi, n; G)$ for $H^m(K(\pi, n); G)$. However, we will keep using the original notation throughout this thesis. We only make this remark because the authors of [MT68] use the simplified notation, which may give rise to confusion.

Finally, we state the classification theorem. Let $\theta \in \mathcal{O}(\pi, n; G, m)$, then $\iota_n \in H^n(K(\pi, n); \pi)$ implies $\theta(\iota_n) \in H^m(K(\pi, n); G)$.

Theorem 2.2.5 (Classification of Cohomology Operations)**.** *There is a one-to-one correspondence*

$$\mathcal{O}(\pi, n; G, m) \longleftrightarrow H^m(K(\pi, n); G)$$

$$\theta \longleftrightarrow \theta(\iota_n).$$

Proof. Let $p\colon \theta \mapsto \theta(\iota_n)$. We will show p has a two-sided inverse.

Let $\phi \in H^m(K(\pi, n); G)$. Consider some $u \in H^n(X; \pi)$ for some arbitrary space X. By Theorem 2.2.1 there is a class $[f] \in [X, K(\pi, n)]$ corresponding to u. Suppose $f : X \to K(\pi, n)$ is a representative of $[f]$. We define a cohomology operation ψ of type $(\pi, n; G, m)$ by $\psi(u) = f^*(\phi) \in H^m(X; G)$. This gives us a map $q\colon H^m(K(\pi, n); G) \to \mathcal{O}(\pi, n; G, m)$ defined by $q\colon \phi \mapsto \psi$.

We can easily verify that q is a two-sided inverse to p. If $X = K(\pi, n)$ and $[f]$ corresponds to ι_n under Theorem 2.2.1, then f is homotopy equivalent to id, thus $\psi(\iota_n) = f^*(\phi) = \phi$. This shows $pq = \text{id}$. Conversely, suppose $\phi = \theta(\iota_n)$. Then $\psi(u) = f^*(\phi) = f^*(\theta(\iota_n)) = \theta(f^*(\iota_n)) = \theta(u)$ for any u. Therefore, $\psi = \theta$ and thus $qp = \text{id}$. This finishes the proof. □

Remark 2.2.6. The above classification theorem is a direct result of the Yoneda lemma. We noted before that $\mathcal{O}(\pi, n; G, m)$ is the natural transformation between $H^n(-; \pi)$ and $H^m(-; G)$, now we can use Theorem 2.2.1 to finish the argument.

Corollary 2.2.7. *There is a one-to-one correspondence*

$$\mathcal{O}(\pi, n; G, m) \longleftrightarrow [K(\pi, n), K(G, m)]$$

Proof. This follows from Theorem 2.2.1 and Theorem 2.2.5 immediately. □

We end this chapter with a final note that the classification theorem we proved above reduces the problem of finding all the cohomology operations of a certain type to the computation of the cohomology of the Eilenberg-MacLane spaces, which will be one of our major objectives in this thesis.

Starting from the next chapter, we will focus on a special family of operations - the Steenrod squares on cohomology with mod 2 coefficients.

Chapter 3

The Steenrod Squares

In this chapter, we construct and examine a specific family of cohomology operations called the Steenrod squares. These cohomology operations were first introduced by Norman Steenrod in 1947 in [Ste47], and they are of type $(\mathbb{Z}/2, n; \mathbb{Z}/2, n+1)$ satisfying a certain set of relations called the Adem relations. There are analogous operations, called the Steenrod reduced pth powers, for $\mathbb{Z}/p$ coefficients with p an odd prime. However, we will only deal with the Steenrod squares in this thesis.

3.1 Construction of the Steenrod Squares

We will follow the classical path in constructing the Steenrod squares, as done in [SE62] and [MT68]. We will first construct a series of related operations called the cup-i products. As the name suggests, the cup-i products can be thought of as a generalization to the familiar cup product from cohomology theory. Our development will be mainly based on the arguments presented in [MT68], but certain changes are made in reference to [Med21].

3.1.1 The Cup-i Products

An important piece in the construction is the acyclic carrier theorem. Before we state the theorem, we need to first make some definitions.

Definition 3.1.1. Let π and G be any groups, and let $\mathbb{Z}[\pi]$ be the group ring of π over $\mathbb{Z}$. Let $K = \{M_i, \partial_i\}$ be a chain complex of free $\mathbb{Z}[\pi]$-modules, and let $\{\sigma_i^\alpha\}$ be a basis for M_i with α ranging over some index set J_i. We say $B = \bigcup_i \{\sigma_i^\alpha\}$ is the *basis* of K. For some $\tau, \sigma \in B$, we denote the coefficient of σ in $\partial\tau$ by $[\tau : \sigma]$.

Let L be another chain complex and suppose G acts on L, and let $h : \pi \to G$ be a group homomorphism.

Definition 3.1.2. An *h-equivariant acyclic carrier $\mathscr{C}$* from K to L, relative to the chosen bases, is a function $\mathscr{C} : B \to \{\text{subcomplexes of } L\}$ that satisfies:

1. $\mathscr{C}(\sigma)$ is acyclic for all $\sigma \in B$;
2. if $[\tau : \sigma] \neq 0$ then $\mathscr{C}(\sigma)$ is a subcomplex of $\mathscr{C}(\tau)$;
3. for any $x \in \pi$ and $\sigma \in B$, we have $h(x) \cdot \mathscr{C}(\sigma)$ is a subcomplex of $\mathscr{C}(\sigma)$.

An h-equivariant chain map $f : K \to L$ is said to be *carried by $\mathscr{C}$* if $f(\sigma) \in \mathscr{C}(\sigma)$ for all $\sigma \in B$.

Topologically, if X is a simplicial complex and $K = C(X)$ is the chain complex corresponding to X, then a basis element of K corresponds to a simplex in X. Condition 2 says if σ is a face of τ, then $\mathscr{C}(\sigma)$ is a subcomplex of $\mathscr{C}(\tau)$. Moreover, f being carried by $\mathscr{C}$ means for every $\sigma \in B$, the chain $f(\sigma)$ is carried by the subcomplex $\mathscr{C}(\sigma)$. We now state the algebraic version of the acyclic carrier theorem.

Theorem 3.1.3 (Acyclic Carrier Theorem)**.** *Let $\mathscr{C}$ be an acyclic carrier from K to L. Let K' be a subcomplex of K, i.e. K' is a free chain complex of $\mathbb{Z}[\pi]$-modules with basis a subset of B. Suppose $f : K' \to L$ is an h-equivariant chain map carried by $\mathscr{C}$, then f extends to a $\mathscr{C}$-carried h-equivariant chain map from K to L. Moreover, the extension is unique up to an h-equivariant chain homotopy carried by $\mathscr{C}$.*

Proof. The proof inducts on the dimension and it is not very enlightening. See either [Mun84, Chapter 1, Section 13] or [MT68, Chapter 2]. □

The acyclic carrier theorem is important to us because it guarantees the existence of our construction below.

We start with the infinite-dimensional sphere S^∞. We can give S^∞ a CW-complex structure as the following. We start with two 0-cells, call them d_0 and Td_0. Then we attach to them two 1-cells with $\pm(d_0 - Td_0)$ as boundaries, call them d_1 and Td_1, respectively. Then we attach to them two 2-cells with $\pm(d_1 + Td_1)$ as boundaries, call them d_2 and Td_2, respectively. We keep doing this for all dimensions. Hence, in each dimension $n \geq 0$, there are two n-cells. Observe, the boundary map on the CW-complex structure acts by $\partial d_i = d_{i-1} + (-1)^i T d_{i-1}$. Also, we clearly have $\partial T = T\partial$ and $T^2 = 1$. Note that T is the flipping map that swaps the two generators in each dimension. It is straight-forward to see that in even dimensions, the only non-zero cycles are generated by $d_{2j} - Td_{2j} = \partial d_{2j+1}$; in odd dimensions, the only non-zero cycles are generated by $d_{2j-1} + Td_{2j-1} = \partial d_{2j}$. As a consequence, we obtain the reduced homology groups of S^∞: $\tilde{H}_i(S^\infty;\mathbb{Z}) \cong \mathbb{Z}/\mathbb{Z} \cong 0$ for all $i \geq 0$. This implies S^∞ is **acyclic**. We let W be the chain complex of S^∞ over $\mathbb{Z}/2$:

$$\cdots \longrightarrow \mathbb{Z}/2^{\oplus 2} \longrightarrow \mathbb{Z}/2^{\oplus 2} \longrightarrow \mathbb{Z}/2^{\oplus 2} \longrightarrow 0.$$

Now, let X be a simplicial complex and $K = C_\bullet(X)$ its chain complex. Let W be the chain complex of S^∞ with $\mathbb{Z}/2$ coefficients as before. Let $\mathbb{Z}/2$ (generated by the flipping map T defined previously) act on the chain complexes $W \otimes K$ and $K \otimes K$ as the following: $T(w\otimes k) = (Tw)\otimes k$ for $w\otimes k \in W\otimes K$, and $T(x\otimes y) = (-1)^{|x|\cdot|y|}(y\otimes x)$ for $x \otimes y \in K \otimes K$. It should be obvious that T^2 indeed acts trivially on these chain complexes. By the Eilenberg-Zilber theorem, there is the Alexander-Whitney map $C_\bullet(X \times X) \xrightarrow{AW} C_\bullet(X) \otimes C_\bullet(X)$, which is a chain-homotopy equivalence. Now we define a map

$$\mathscr{C} : W \otimes K \longrightarrow \{\text{subcomplexes of } K \otimes K\}$$

$$d_i \otimes \sigma \longmapsto AW(C_\bullet(\sigma \times \sigma)),$$

where d_i is a simplex in S^∞ and thus a basis element of W, and σ is some basis element of K. Observe that $d_i \otimes \sigma$ is a face of $d_j \otimes \tau$ only if σ is a face of τ, hence it follows

$\mathscr{C}(d_i \otimes \sigma)$ is a subcomplex of $\mathscr{C}(d_j \otimes \tau)$. Moreover, σ is some simplex in X and thus contractible, this then implies $\sigma \times \sigma$ is also contractible. Hence, $C_\bullet(\sigma \times \sigma)$ is acyclic, and since AW induces isomorphism between homology groups, we see $\mathscr{C}(d_i \otimes \sigma)$ is acyclic for every basis element $d_i \otimes \sigma$. Finally, since T acting on an element in $K \otimes K$ does not change its degree, it follows that $T\mathscr{C}(d_i \otimes \sigma) \subseteq \mathscr{C}(d_i \otimes \sigma)$ for any $d_i \otimes \sigma$. Therefore, we conclude that $\mathscr{C}$ is an h-equivariant acyclic carrier, where h is the identity map. It then follows from the acyclic carrier theorem that there exists an h-equivariant chain map $\Delta : W \otimes K \to K \otimes K$ carried by $\mathscr{C}$.

Let us examine this Δ in greater details. Consider the restrictions $\Delta_i = \Delta|_{d_i \otimes K}$. The first of them, Δ_0, can be regarded as a map from K to $K \otimes K$. Indeed, since d_0 is a single element, we have an obvious isomorphism $d_0 \otimes K \cong K$. By our construction, Δ_0 is carried by the diagonal carrier, and thus we can use it to compute the cup products in K. Now, let's look at $T\Delta_0$. Because Δ is h-equivariant, we have that $T\Delta_0(\sigma) = T\Delta(d_0 \otimes \sigma) = \Delta T(d_0 \otimes \sigma) = \Delta(Td_0 \otimes \sigma)$, where the last equality holds by how we defined T to act on $W \otimes k$. Just like Δ_0, this $T\Delta_0$ can also be regarded as a map $K \to K \otimes K$, also carried by the diagonal carrier. By the acyclic carrier theorem, this implies Δ_0 and $T\Delta_0$ are equivariantly homotopic (i.e. there is an h-equivariant chain homotopy $\Delta_0 \simeq T\Delta_0$ carried by the diagonal carrier). In fact, we shall prove the following result.

Proposition 3.1.4. *For all $i \geq 1$, Δ_i and Δ_{i-1} satisfy the following relation.*

$$\partial_{K\otimes K}\Delta_i - (-1)^i \Delta_i \partial_K = \Delta_{i-1} + (-1)^i T\Delta_{i-1}. \tag{3.1.1}$$

Proof. I will omit the subscripts for the boundary maps, as this should raise no confusion. Recall, $\partial d_i = d_{i-1} + (-1)^i T d_{i-1}$, and

$$\partial(d_i \otimes \sigma) = \partial d_i \otimes \sigma + (-1)^i d_i \otimes \partial\sigma.$$

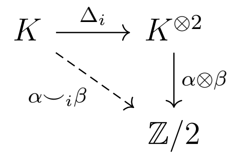
Acting Δ on both sides, the LHS is equal to $\Delta(\partial(d_i \otimes \sigma)) = \partial(\Delta(d_i \otimes \sigma)) = \partial\Delta_i(\sigma)$, where the first equality holds because Δ is a chain map. On the other hand, the RHS is equal to

$$\begin{aligned}\Delta(\partial d_i \otimes \sigma) + (-1)^i \Delta(d_i \otimes \partial\sigma) =& \Delta((d_{i-1} + (-1)^i T d_{i-1}) \otimes \sigma) + (-1)^i \Delta_i(\partial\sigma) \\ =& \Delta(d_{i-1} \otimes \sigma) + (-1)^i \Delta(T d_{i-1} \otimes \sigma) + (-1)^i \Delta_i \partial(\sigma) \\ =& \Delta_{i-1}(\sigma) + (-1)^i T\Delta_{i-1}(\sigma) + (-1)^i \Delta_i \partial(\sigma).\end{aligned}$$

Moving the third term above to the LHS ($= \partial\Delta_i(\sigma)$) yields the desired equation. □

Taking coefficients in $\mathbb{Z}/2$, Eq.(3.1.1) turns into the form $\Delta_{i-1} - T\Delta_{i-1} = \partial\Delta_i + \Delta_i\partial$. This is saying that each Δ_i is a chain homotopy between Δ_{i-1} and $T\Delta_{i-1}$.

Definition 3.1.5. The above construction is called a *cup-i construction*, and the maps Δ_i are called the *cup-i coproducts*. The *cup-i products*, denoted $\smile_i$, are taken to be the linear dual of Δ_i. That is, given two cochains α and β and a chain c, we define $\alpha \smile_i \beta$ to be the map that makes the following diagram commute.



Specifically, with correct dimensions, we have

$$\smile_i\colon C^p(X) \otimes C^q(X) \longrightarrow C^{p+q-i}(X)$$

$$\alpha \otimes \beta \longmapsto \alpha \smile_i \beta,$$

where $(\alpha \smile_i \beta)(c) = (\alpha \otimes \beta)\Delta_i(c) = (\alpha \otimes \beta)\Delta(d_i \otimes c)$, for some $c \in C_{p+q-i}(X)$.

It may now appear that the cup-i products depend on an explicit choice of Δ, but we will later show that this choice in fact makes no difference. Let us now look at how the cup-i products interact with the coboundary map.

Proposition 3.1.6 (Coboundary Formula)**.** *We have*

$$\delta(\alpha \smile_i \beta) = (-1)^i \delta\alpha \smile_i \beta + (-1)^{i+p}\alpha \smile_i \delta\beta - (-1)^i \alpha \smile_{i-1} \beta - (-1)^{pq}\alpha \smile_{i-1} \beta, \tag{3.1.2}$$

with the convention that $\alpha \smile_{-1} \beta = 0$.

Proof sketch. Let $c \in C_{p+q-i+1}(X)$, then $(\delta(\alpha \smile_i \beta))(c) = (\alpha \smile_i \beta)(\partial c) = (\alpha \otimes \beta)\Delta(d_i \otimes \partial c)$. Rewriting $d_i \otimes \partial c = (-1)^i \partial(d_i \otimes c) - (-1)^i \partial d_i \otimes c$ (recall the boundary formula for tensor products) and using the equation $\partial d_i = d_{i-1} + (-1)^i T d_{i-1}$, we can deduce

$$\begin{aligned}(\delta(\alpha \smile_i \beta))(c) = (-1)^i \delta(\alpha \otimes \beta)\Delta(d_i \otimes c) - (-1)^i (\alpha \otimes \beta)\Delta(d_{i-1} \otimes c) \\ - (-1)^{pq}(\alpha \otimes \beta)\Delta(d_{i-1} \otimes c).\end{aligned}$$

Now using the coboundary formula for tensor products to rewrite the first term on the RHS, we then obtain the desired formula. $\square$

3.1.2 The Steenrod Squares

We are now ready to define the Steenrod squares in terms of these cup-i products. Suppose $\alpha \in C^p(X)$ is a cocycle mod 2, i.e. $\delta\alpha = 2\gamma$ for some $\gamma \in C^{p+1}(X)$. Plugging $\alpha \smile_i \alpha$ into the coboundary formula Eq.(3.1.2), we obtain

$$\begin{aligned}\delta(\alpha \smile_i \alpha) =&(-1)^i\delta\alpha \smile_i \alpha + (-1)^{i+p}\alpha \smile_i \delta\alpha - (-1)^i\alpha \smile_{i-1} \alpha - (-1)^{pq}\alpha \smile_{i-1} \alpha\\ =&(-1)^i 2\gamma \smile_i \alpha + (-1)^{i+p}\alpha \smile_i 2\gamma - ((-1)^i + (-1)^{pq})\alpha \smile_i \alpha.\end{aligned}$$

However, from the definition of the cup-i product, we can readily observe

$$(2\gamma \smile_i \alpha)(c) = (2\gamma \otimes \alpha)\Delta_i(c) = 2(\gamma \otimes \alpha)\Delta_i(c) = 2(\gamma \smile_i \alpha)(c).$$

Hence, we have $2\gamma \smile_i \alpha = 2(\gamma \smile_i \alpha)$, and similarly $\alpha \smile_i 2\gamma = 2(\alpha \smile_i \gamma)$. Therefore,

$$\begin{aligned}\delta(\alpha \smile_i \alpha) =&(-1)^i 2(\gamma \smile_i \alpha) + (-1)^{i+p}2(\alpha \smile_i \gamma) - ((-1)^i + (-1)^{pq})\alpha \smile_i \alpha\\ \equiv& - ((-1)^i + (-1)^{pq})\alpha \smile_i \alpha \pmod 2\\ \equiv& 0 \pmod 2,\end{aligned}$$

where the last equivalence holds because $(-1)^i + (-1)^{pq}$ is either 0 or ± 2. Hence, $\alpha \smile_i \alpha$ is also a cocycle mod 2. Since "squaring" under the cup-i product preserves cocycles, we can define the following maps:

$$\begin{aligned}Sq_i \colon Z^p(X;\mathbb{Z}/2) &\longrightarrow Z^{2p-i}(X;\mathbb{Z}/2)\\ \alpha &\longmapsto \alpha \smile_i \alpha.\end{aligned}$$

Moreover, the projection $Sq_i : Z^p(X;\mathbb{Z}/2) \to H^{2p-i}(X;\mathbb{Z}/2)$ of cocycles onto cohomology classes is a group homomorphism. The proof goes as the following. Suppose $c \in C_{2p-i}(X)$ and $\alpha, \beta \in Z^p(X;\mathbb{Z}/2)$, then we have

$$\begin{aligned} Sq_i(\alpha+\beta)(c) =& ((\alpha+\beta) \smile_i (\alpha+\beta))(c) = ((\alpha+\beta)\otimes(\alpha+\beta))\Delta(d_i \otimes c) \\ =& (\alpha\otimes\alpha + \alpha\otimes\beta + \beta\otimes\alpha + \beta\otimes\beta)\Delta(d_i\otimes c) \\ =& Sq_i(\alpha)(c) + Sq_i(\beta)(c) + (\alpha \smile_i \beta + \beta \smile_i \alpha)(c) \end{aligned}$$

But observe that using the coboundary formula Eq.(3.1.2), we obtain $\delta(\alpha \smile_{i+1} \beta) \equiv \alpha \smile_i \beta + \beta \smile_i \alpha \pmod 2$. Therefore, the sum of the cross terms above is a coboundary, hence it vanishes in $H^{2p-i}(X;\mathbb{Z}/2)$. We thus get $Sq_i(\alpha+\beta) = Sq_i(\alpha) + Sq_i(\beta)$, as desired.

It should also be noted that Sq_i preserves coboundaries. Let α be a coboundary, i.e. $\alpha = \delta\beta$ for some cochain β, then we have $Sq_i(\alpha) \equiv \delta(\beta \smile_i \alpha + \beta \smile_{i-1} \beta) \pmod 2$. The verification is just a routine computation using the coboundary formula Eq.(3.1.2), and we omit it. As an immediate consequence, we have the following result.

Proposition 3.1.7. *The map Sq_i defined as above passes to a homomorphism:*

$$Sq_i : H^p(X;\mathbb{Z}/2) \longrightarrow H^{2p-i}(X;\mathbb{Z}/2).$$

Proposition 3.1.8. *Moreover, let $f : X \to Y$ be a continuous map, so that it induces a homomorphism $f^* : H^i(Y;\mathbb{Z}/2) \to H^i(X;\mathbb{Z}/2)$. Then Sq_i commutes with f^* as in the following diagram.*

$$\begin{array}{ccc} H^p(Y;\mathbb{Z}/2) & \xrightarrow{Sq_i} & H^{2p-i}(Y;\mathbb{Z}/2) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^p(X;\mathbb{Z}/2) & \xrightarrow{Sq_i} & H^{2p-i}(X;\mathbb{Z}/2) \end{array}$$

Proof. It suffices to consider f simplicial. Let α be a p-cochain of Y. Observe that we have the following formulae:

$$f^*(Sq_i(a)) : c \longmapsto (\alpha\otimes\alpha)\Delta^Y(d_i\otimes f(c)) = (\alpha\otimes\alpha)\Delta^Y(1\otimes f)(d_i\otimes c)$$

$$Sq_i(f^*(\alpha)) : c \longmapsto (f^*(\alpha)\otimes f^*(\alpha))\Delta^X(d_i\otimes c) = (\alpha\otimes\alpha)(f\otimes f)\Delta^X(d_i\otimes c).$$

Observe now

$$\begin{array}{llcccccc} & \Delta^Y(1\otimes f): & W\otimes K & \longrightarrow & W\otimes L & \longrightarrow & L\otimes L \\ & & d_i\otimes c & \longmapsto & d_i\otimes f(c) & \longmapsto & f(c)\otimes f(c), \\ \text{and} & (f\otimes f)\Delta^X: & W\otimes K & \longrightarrow & K\otimes K & \longrightarrow & L\otimes L \\ & & d_i\otimes c & \longmapsto & c\otimes c & \longmapsto & f(c)\otimes f(c). \end{array}$$

Note c is a $(2p-i)$-chain in X. Therefore, both $\Delta^Y(1\otimes f)$ and $(f\otimes f)\Delta^X$ are carried by the acyclic carrier $\mathscr{C}: W\otimes K\to L\otimes L$, which is given by $\mathscr{C}(d_i\otimes\sigma) = AW(C_\bullet(f(\sigma)\times f(\sigma)))$. By the acyclic carrier theorem, these two chain maps must be equivariantly chain homotopic. Hence, we conclude that the images of α under f^*Sq_i and Sq_if^* are cohomologous (i.e. they differ by a coboundary), and thus $Sq_if^* = f^*Sq_i$. □

As a corollary, this addresses an issue we raised immediately after the definition of the cup-i products.

Corollary 3.1.9. *The operation Sq_i is independent of the choice of Δ.*

Proof. Let $X = Y$ in Proposition 3.1.8, and let Δ^X, Δ^Y be two different choices of Δ. Let $f : X \to Y$ be the identity map, which induces isomorphisms on cohomology groups. Then Proposition 3.1.8 shows $Sq_i^X = Sq_i f^* = f^* Sq_i = Sq_i^Y$. □

At the long last, we are finally ready to define the Steenrod squares.

Definition 3.1.10 (Steenrod Squares)**.** Denote by Sq^i the homomorphisms

$$Sq^i : H^p(X;\mathbb{Z}/2) \longrightarrow H^{p+i}(X;\mathbb{Z}/2),$$

given by $Sq^i = Sq_{p-i}$, where i ranges over $0, 1, \ldots, p$. If $i \notin \{0, 1, \ldots, p\}$, we then take the convention $Sq^i = 0$.

Remark 3.1.11. Sq^i raises the cohomology by degree i.

To end this section, let us consider the Steenrod squares for relative cohomology. Let L be a subcomplex of K. Note that we have the short exact sequence at the cochain level:

$$0 \longrightarrow C^\bullet(K, L) \xrightarrow{q^*} C^\bullet(K) \xrightarrow{j^*} C^\bullet(L) \longrightarrow 0.$$

Note that we can assume $\Delta^L = \Delta^K|_{W\otimes L}$, because $\Delta^K(d_i \otimes \sigma) \in C_\bullet(\sigma \otimes \sigma) \subseteq L \otimes L$, for any $\sigma \in L$. It turns out that $j^*(\alpha \smile_i \beta) = j^*(\alpha) \smile_i j^*(\beta)$ for $\alpha, \beta \in C^i(K)$. Therefore, we can define the cup-i products as follows. Let $\alpha, \beta \in C^\bullet(K, L)$, then by exactness and the above note, we have

$$j^*(q^*(\alpha) \smile_i q^*(\beta)) = j^*q^*(\alpha) \smile_i j^*q^*(\beta) = 0.$$

Hence, by exactness, this implies $q^*(\alpha) \smile_i q^*(\beta) \in \ker j^* = \operatorname{im} q^*$. But q^* is injective, thus we can define $\alpha \smile_i \beta$ to be the unique cochain in $C^\bullet(K, L)$ such that $q^*(\alpha \smile_i \beta) = q^*(\alpha) \smile_i q^*(\beta)$. The coboundary formula we had before easily carries over. Therefore, all the remarks about cocycle-preserving and coboundary-preserving properties also carry over, and we have homomorphisms

$$Sq^i : H^p(K, L; \mathbb{Z}/2) \longrightarrow H^{p+i}(K, L; \mathbb{Z}/2).$$

It is obvious that we have $q^* Sq^i = Sq^i q^*$.

Recall, in the long exact sequence of cohomology, we have the connecting homomorphism $\delta^* : H^p(L; \mathbb{Z}/2) \to H^{p+1}(K, L; \mathbb{Z}/2)$, which can be obtained through a diagram chase. We have the following.

Proposition 3.1.12. *Sq^i commutes with the connecting homomorphism δ^* as in the following commutative diagram.*

$$\begin{array}{ccc} H^p(L; \mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{p+i}(L; \mathbb{Z}/2) \\ \delta^* \downarrow & & \downarrow \delta^* \\ H^{p+1}(K, L; \mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{p+i+1}(K, L; \mathbb{Z}/2) \end{array}$$

Proof. See [MT68, Chapter 2]. □

Finally, we remark that the Steenrod squares are also compatible with suspension. Given a space X, we define the suspension SX of X to be the quotient of $X \times I$ obtained from identifying $X \times \{0\}$ as a single point and $X \times \{1\}$ as another point. In reduced cohomology, S leads to the *suspension isomorphism* $S^* : \tilde{H}^p(X) \to \tilde{H}^{p+1}(SX)$

through the following composition,

$$\tilde{H}^p(X) \longrightarrow H^{p+1}(CX, X) \longrightarrow \tilde{H}^{p+1}(SX),$$

where CX is the cone over X. In the above composition, the first isomorphism is given by the connecting homomorphism, and the second isomorphism is given by excision. We have the following.

Proposition 3.1.13. *Sq^i commutes with suspension S^* as in the following commutative diagram.*

$$\begin{array}{ccc} \tilde{H}^p(X) & \xrightarrow{Sq^i} & \tilde{H}^{p+i}(X) \\ {\scriptstyle S^*}\downarrow & & \downarrow{\scriptstyle S^*} \\ \tilde{H}^{p+1}(SX) & \xrightarrow{Sq^i} & \tilde{H}^{p+i+1}(SX) \end{array}$$

Proof. It follows from the naturality of the Steenrod squares and Proposition 3.1.12.

□

3.2 Properties of the Steenrod Squares

In this section, we discuss some important properties of the Steenrod squares.

Theorem 3.2.1. *The operations Sq^i $(i \geq 0)$ have the following properties:*

1. Sq^i is a natural homomorphism $H^p(K, L; \mathbb{Z}/2) \to H^{p+i}(K, L; \mathbb{Z}/2)$;

2. If $i > p$, then $Sq^i(x) = 0$ for all $x \in H^p(K, L; \mathbb{Z}/2)$;

3. $Sq^i(x) = x^2$ for all $x \in H^i(K, L; \mathbb{Z}/2)$;

4. Sq^0 is the identity homomorphism;

5. Sq^1 is the Bockstein homomorphism δ_2;

6. $\delta^* Sq^i = Sq^i \delta^*$, where $\delta^* : H^p(L; \mathbb{Z}/2) \to H^{p+1}(K, L; \mathbb{Z}/2)$ is the connecting homomorphism;

7. Cartan formula:

$$Sq^i(xy) = \sum_j Sq^j(x) Sq^{i-j}(y);$$

8. Adem relations: for $a < 2b$,

$$Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c,$$

where the binomial coefficient is taken mod 2.

Remark 3.2.2. These properties can be taken as axioms that completely characterize the Steenrod squares. See [SE62].

In the previous section, we already showed properties 1 and 6. Note that property 2 is a direct consequence of our definition of the Steenrod squares. Therefore, in this section, we will prove properties 3, 4, 5, 7, and 8.

Proof of Property 3. Recall by definition, if x is a cochain of dimension i, then $Sq^i(x) = Sq_0(x)$. But $Sq_0(x) = x \smile_0 x$ is defined by $(x \smile_0 x)(c) = (x \otimes x)(\Delta_0(c))$, where c is some chain of dimension i and $\Delta_0 = \Delta|_{d_0 \otimes K}$ is the restriction of the chain map Δ we defined in the previous section. As we noted before, Δ_0 is carried by the diagonal carrier, and thus it is a chain approximation to the diagonal map, which implies $(x \otimes x)(\Delta_0(c)) = (x \smile x)(c)$. Therefore, $Sq^i(x) = x \smile x = x^2$. □

Next, we will prove properties 4 and 5 together. Before we proceed, we need to define a Bockstein homomorphism.

Definition 3.2.3. Given a short exact sequence $0 \to G \to L \to K \to 0$ of abelian groups, it induces a short exact sequence of chain complexes

$$0 \longrightarrow C^{\bullet}(X;G) \longrightarrow C^{\bullet}(X;L) \longrightarrow C^{\bullet}(X;K) \longrightarrow 0,$$

which then leads to an associated long exact sequence

$$\cdots \longrightarrow H^n(X;G) \longrightarrow H^n(X;L) \longrightarrow H^n(X;K) \longrightarrow H^{n+1}(X;G) \longrightarrow \dots$$

where X is any space. The connecting homomorphism $H^n(X;K) \to H^{n+1}(X;G)$ is called a *Bockstein homomorphism.*

Note that the associated long exact sequence exists for relative cohomology as well. In property 5, δ_2 is the Bockstein homomorphism corresponding to the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. There is another Bockstein homomorphism that shows up frequently, namely the Bockstein homomorphism β corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$. This $\beta : H^p(K,L;\mathbb{Z}/2) \to H^{p+1}(K,L;\mathbb{Z})$ is defined as the following. Take $x \in H^p(K,L;\mathbb{Z})$ represented by some cocycle c, choose an integral cochain c' that maps to c under reduction mod 2, then $\delta c = 0$ implies $\delta c' \equiv 0 \pmod 2$ and thus $\delta c'$ is divisible by 2, we let $\frac{1}{2}\delta c'$ represent βx.

Remark 3.2.4. δ_2 is the composition of β and the reduction homomorphism.

The reason we prove properties 4 and 5 together is the following lemma.

Lemma 3.2.5. *The composition $\delta_2 Sq^i = 0$ if i is odd, and $\delta_2 Sq^i = Sq^{i+1}$ if i is even.*

Proof. Given $u \in H^p(K, L; \mathbb{Z}/2)$, represent it by a cocycle c'. Let c be an integral cochain that maps to c' under reduction mod 2. By definition, $Sq^i(u) = Sq_{p-i}(u) = [c \smile_{p-i} c]$, where the last term is the class represented by $c \smile_{p-i} c$ reduced mod 2. Since c' is a cocycle, $\delta c' = 0$. It then follows that $\delta c \equiv 0 \pmod 2$. Hence, $\delta c = 2a$ for some integral cochain $a \in C^{p+1}(K, L)$. Let $j = p - i$, then by the coboundary formula Eq.(3.1.2),

$$\delta(c \smile_j c) = (-1)^j 2a \smile_j c + (-1)^i c \smile_j 2a - (-1)^j c \smile_{j-1} c - (-1)^p c \smile_{j-1} c.$$

By the above remark, we then have

$$\begin{aligned}
\delta_2(Sq^i(u)) = \delta_2([c \smile_{p-i} c]) &= \left[\frac{1}{2}\delta(c \smile_j c)\right] \\
&= \left[(-1)^j a \smile_j c + (-1)^i c \smile_j a - ((-1)^j + (-1)^p) c \smile_{j-1} c\right] \\
&\equiv [\delta(c \smile_{j+1} a)] + (s)[c \smile_{j-1} c],
\end{aligned}$$

where in the last line, the first term is a class of a coboundary and hence it disappears, and the coefficient s of the second term is 0 if i is odd and 1 if i is even. Now observe that $[c \smile_{j-1} c] = Sq_{p-i-1}(u) = Sq^{i+1}(u)$. Plugging it back to the equation above yields the desired result. $\square$

In the special case where $i = 0$, this lemma tells us $\delta_2 Sq^0 = Sq^1$. Therefore, it remains to prove property 4, and property 5 will be a direct consequence of property 4.

Proof of Properties 4 and 5. We prove property 4 by a sequence of generalization. We start with the real projective plane $\mathbb{R}P^2$. Let α be the one-dimensional generator of the cohomology ring of $\mathbb{R}P^2$ with $\mathbb{Z}/2$ coefficients, then $\delta_2 Sq^0(\alpha) = Sq^1(\alpha) = \alpha^2 \neq 0$, which implies $Sq^0(\alpha) \neq 0$. Hence, we must have $Sq^0(\alpha) = \alpha$ because α is the only non-zero element in $H^1(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$. This shows property 4 holds for $\mathbb{R}P^2$.

Now consider S^1. Take a map $f\colon S^1 \to \mathbb{R}P^2$ such that $f^*(\alpha) = \sigma$, where α is the same as above and σ is the generator of the cohomology ring of S^1 with $\mathbb{Z}/2$ coefficients. By the naturality condition, we have

$$Sq^0(\sigma) = Sq^0(f^*(\alpha)) = f^*(Sq^0(\alpha)) = f^*(\alpha) = \sigma.$$

Thus, property 4 holds for S^1.

Recall that we can obtain S^{n+1} from S^n by suspension, therefore by Proposition 3.1.13 and the above result for S^1, we easily see that property 4 holds for all S^n.

Next, let K be a complex of dimension n. By the Hopf-Whitney theorem, we can map K to S^n so that σ pulls back to any given class in $H^n(K; \mathbb{Z}/2)$. It then follows that property 4 holds for K by commutativity. Now let K be any complex and n any non-negative integer, then the inclusion $j\colon K^n \hookrightarrow K$ of the n-skeleton K^n into K induces a monomorphism $j^*\colon H^n(K; \mathbb{Z}/2) \to H^n(K^n; \mathbb{Z}/2)$. Again by naturality, property 4 must hold for any n-dimensional cohomology class in K. Therefore, property 4 holds for absolute cohomology.

Finally, let (K, L) be a pair and consider the space $K \cup_L CL$, namely the space obtained by attaching the cone over L to K at the common subspace L. Then we have

$$H^n(K, L) \cong H^n(K \cup_L CL, CL) \cong \tilde{H}^n(K \cup_L CL),$$

where the first isomorphism follows from excision and the second follows from the simple fact that CL is contractible. The composed isomorphism $H^n(k, L) \to \tilde{H}^n(K \cup_L CL)$ commutes with Sq^0 by naturality, thus property 4 holds for relative cohomology.

We have thus proved property 4, and property 5 follows as an immediate consequence. □

Next, let us look at the Cartan formula (property 7). We will first show the Cartan formula holds when xy is interpreted as the direct product $x \times y$. From this, we will then deduce that the Cartan formula also holds when xy is interpreted as the cup product $x \smile y$.

Proof of the Cartan Formula (Property 7). Let W be the chain complex of S^∞ as before. Let K and L be two arbitrary chain complexes. Consider the composition

$$\begin{array}{ccccc} W \otimes K \otimes L & \xrightarrow{r \otimes \mathrm{id} \otimes \mathrm{id}} & W \otimes W \otimes K \otimes L & \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} & W \otimes K \otimes W \otimes L \\ & & & & \downarrow{\scriptstyle \Delta_K \otimes \Delta_L} \\ & & & & K \otimes K \otimes L \otimes L \\ & & & & \downarrow{\scriptstyle \mathrm{id} \otimes \tau \otimes \mathrm{id}} \\ & & & & K \otimes L \otimes K \otimes L, \end{array}$$

where τ is the flipping map, i.e. $\tau(a \otimes b) = b \otimes a$, and the map $r \colon W \to W \otimes W$ is defined by

$$r(d_i) = \sum_{0 \le j \le i} (-1)^{j(i-j)} d_j \otimes T^j d_{i-j}$$

$$r(Td_i) = T(r(d_i)),$$

with T and d_i as defined in the previous section. We shall not be concerned with the sign, since our result is in $\mathbb{Z}/2$ coefficients. Denote the above composition by $\Delta_{K\otimes L}$, then we can use it to compute Sq^i in $K \otimes L$.

Let u be a cochain in K, v a cochain in L, a a chain in K, and b a chain in L. Let $p = \dim u$, $q = \dim v$, and $n = p + q - i$. Then we have the following,

$$\begin{aligned}
Sq^i(u \times v)(a \otimes b) &= ((u \otimes v) \smile_n (u \otimes v))(a \otimes b) \\
&= (u \otimes v \otimes u \otimes v)\Delta_{K\otimes L}(d_n \otimes a \otimes b) \\
&= (u \otimes u \otimes v \otimes v) \sum \Delta_K(d_j \otimes a) \otimes T^j \Delta_L(d_{n-j} \otimes b) \\
&= \sum (u \smile_j u)(a) \otimes (v \smile_{n-j} v)(b) \\
&= \sum Sq^{p-j}(u)(a) \otimes Sq^{q-n+j}(v)(b) \\
&= \sum (Sq^{p-j}(u) \times Sq^{q-n+j}(b))(a \otimes b),
\end{aligned}$$

where in the third line, T^j is either 1 or -1. But since we are considering $\mathbb{Z}/2$ coefficients, T^j is in fact always 1. Now recall by definition, $Sq^i(x) = 0$ if i is not within the range $0 \leq i \leq \dim x$. Hence,

$$\begin{aligned}
Sq^i(u \times v) &= \sum_{j'=0}^{n} Sq^{p-j'}(u) \times Sq^{q-n+j'}(v) \\
&= \sum_{j=i-q}^{p} Sq^j(u) \times Sq^{i-j}(v) \qquad \leftarrow \text{let } j = p - j' \\
&= \sum_{j=0}^{i} Sq^j(u) \times Sq^{i-j}(v).
\end{aligned}$$

This proves the Cartan formula in the interpretation of direct product, we will now see it also holds in the interpretation of cup product. Indeed, if φ is the diagonal map of K, then for any $x, y \in H^*(K; \mathbb{Z}/2)$ we have $x \smile y = \varphi^*(x \times y)$, and thus

$$\begin{aligned}
Sq^i(x \smile y) &= Sq^i \varphi^*(x \times y) \\
&= \varphi^* Sq^i(x \times y) \\
&= \varphi^* \sum_{j=0}^{i} Sq^j(x) \times Sq^{i-j}(y) \\
&= \sum_{j} Sq^j(x) \smile Sq^{i-j}(y).
\end{aligned}$$

Hence, we prove the Cartan formula in the interpretation of cup product. □

We observe that the Cartan formula shows that the squares Sq^i are not ring homomorphisms. However, we can combine them together to form one.

Definition 3.2.6. We define the map $Sq\colon H^*(K; \mathbb{Z}/2) \to H^*(K; \mathbb{Z}/2)$ by

$$Sq(u) = \sum_{i} Sq^i(u).$$

Remark 3.2.7. The sum is always finite by the definition of the squares. Also, Sq^i is defined on non-homogeneous elements in $H^*(K;\mathbb{Z}/2)$ by requiring it to be compatible with addition.

Proposition 3.2.8. *Sq is a ring homomorphism.*

Proof. Let $u, v \in H^*(K;\mathbb{Z}/2)$. By definition, we have

$$Sq(u) \smile Sq(v) = \sum_i Sq^i(u) \smile \sum_i Sq^i(v).$$

Expanding this cup product by linearity, we then have that at each dimension $\dim u + \dim v + j$, there is a term $\sum_i Sq^i(u) \smile Sq^{j-i}(v)$. Summing over all j, we then have

$$\begin{aligned} Sq(u) \smile Sq(v) &= \sum_i Sq^i(u) \smile \sum_i Sq^i(v) \\ &= \sum_j \sum_i Sq^i(u) \smile Sq^{j-i}(v) \\ &= \sum_j Sq^j(u \smile v) \\ &= Sq(u \smile v), \end{aligned}$$

where the third equality follows from the Cartan formula, and the last equality follows from the definition of Sq. Therefore, Sq is a ring homomorphism. $\square$

The next proposition is a direct application of the above result.

Proposition 3.2.9. *Given some $u \in H^1(K;\mathbb{Z}/2)$, we have $Sq^i(u^j) = \begin{pmatrix} j \\ i \end{pmatrix} u^{j+i}$.*

Proof. Since u is a one-dimensional cohomology class, by properties 2 to 4 we have $Sq(u) = Sq^0(u) + Sq^1(u) = u + u^2$. But Sq is a ring homomorphism, hence $Sq(u^j) =$

$(u+u^2)^j = u^j \sum_k \begin{pmatrix} j \\ k \end{pmatrix} u^k$. Therefore, $Sq^i(u^j) = u^j \begin{pmatrix} j \\ i \end{pmatrix} u^i = \begin{pmatrix} j \\ i \end{pmatrix} u^{j+i}$. $\square$

Lastly, we will prove the Adem relations (property 8). We will not present the full proof here, as some part of the proof is elementary but tedious (it involves a lengthy exercise in combinatorics and the Cartan formula), and some other part requires a result from Chapter 6 (namely, Serre's theorem). We start by defining the operators

$$R\colon = Sq^a Sq^b + \sum_{c=0}^{\lfloor a/2 \rfloor} \begin{pmatrix} b-c-1 \\ a-2c \end{pmatrix} Sq^{a+b-c} Sq^c.$$

Therefore, to show the Adem relations hold, we need to show $R \equiv 0 \pmod 2$ for every R.

Lemma 3.2.10. *Let y be a fixed cohomology class such that $R(y) = 0$ for every R, then $R(xy) = 0$ for every one-dimensional cohomology class x (and for every R).*

Proof. See [MT68, Chapter 3]. $\square$

Let K_n denote the topological product of n copies of $K(\mathbb{Z}/2, 1)$. We know the cohomology of $K(\mathbb{Z}/2, 1)$, since $\mathbb{R}P^\infty$ is a model of it. Specifically, $H^*(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$ is the polynomial ring $\mathbb{Z}/2[\alpha]$ on one generator α, which is a one-dimensional cohomology class. This result can be found, for example, in [MT68, Chapter 1]. Therefore, by the Künneth theorem, $H^*(K_n; \mathbb{Z}/2)$ is the polynomial ring $\mathbb{Z}/2[\alpha_1, \ldots, \alpha_n]$, where α_i corresponds to the non-trivial one-dimensional cohomology class in the i^{th} copy of $K(\mathbb{Z}/2, 1)$. Recall from basic algebra, $\mathbb{Z}/2[\alpha_1, \ldots, \alpha_n]$ has the subring of symmetric polynomials $\mathbb{Z}/2[\sigma_1, \ldots, \sigma_n]$, where σ_i is the *elementary symmetric function of degree* i. For example, $\sigma_1 = x_1 + \cdots + x_n$.

Lemma 3.2.11. *For every R and for every $n \geq 1$, $R(\sigma_n) = 0$.*

Proof. Denote the ring unit in $H^*(K_n; \mathbb{Z}/2)$ by 1, then by property 2, we have $R(1) = 0$ for all R. Then by Lemma 3.2.10, $R(\alpha_i) = R(1\alpha_i) = R(1)R(\alpha_i) = 0$ for all i and for all R. By induction on n, we then have $R(\sigma_n) = 0$ for all $n \geq 1$ and for all R, which again follows from Lemma 3.2.10. □

Lemma 3.2.12. *Assuming $\mathbb{Z}/2$ coefficients, let K be any space, let y be any n-dimensional cohomology class of K, and let $R_{a,b}$ be the Adem relation for $Sq^a Sq^b$ with $a + b \leq n$, then $R(y) = 0$.*

Proof sketch. As a consequence of Serre's theorem (Theorem 6.3.5), the map $f\colon K_n \to K(\mathbb{Z}/2, n)$ whose induced map f^* takes ι_n to σ_n is a monomorphism through dimension $2n$. By Lemma 3.2.11, $R_{a,b}(\sigma_n) = 0$. Also, $R_{a,b}(\iota_n)$ has dimension $n + a + b \leq 2n$, thus by the f^* above we must have $R_{a,b}(\iota_n) = 0$. Take a map $g\colon K \to K(\mathbb{Z}/2, n)$ such that $g^*(\iota_n) = y$, the desired result then follows from naturality. □

Lemma 3.2.13. *For an arbitrary R, if $R(y) = 0$ for every n-dimensional cohomology class y, then $R(z) = 0$ for every $(n-1)$-dimensional cohomology class z.*

Proof. Let u be the generator of $H^1(S^1; \mathbb{Z}/2)$, then property 2 implies $Sq^i(u) = 0$ for all $i > 1$ and property 3 implies $Sq^1(u) = u^2 = 0$. Hence, $Sq^i(u) = 0$ for all

$i > 0$. By the Cartan formula, $R(u \times z) = u \times R(z)$. But $R(u \times z) = 0$ by the above observation and the fact that $u \times z$ has dimension n. Hence, $u \times R(z) = 0$, which implies $R(z) = 0$. □

Proof sketch of the Adem Relations (Property 8). It follows directly from Lemma 3.2.12 and Lemma 3.2.13. We use (downward) induction on dimension: Lemma 3.2.12 shows $R_{a,b}(y) = 0$ for all cohomology classes y in any space K with dimension $n \geq a+b$. That is, $R_{a,b} = 0$ for all dimensions $n \geq a+b$. Use downward induction starting from dimension $a+b$, Lemma 3.2.13 then shows that $R_{a,b} = 0$ for all dimensions $n \geq 0$. □

Example 3.2.14. Some Adem relations:

$Sq^1 Sq^{2n+1} = 0$

$Sq^1 Sq^{2n} = Sq^{2n+1}$

$Sq^2 Sq^{4n-2} = Sq^{4n-1} Sq^1$

$Sq^2 Sq^{4n-1} = Sq^{4n+1} + Sq^{4n} Sq^1$

$Sq^2 Sq^{4n} = Sq^{4n+2} + Sq^{4n+1} Sq^1$

$Sq^2 Sq^{4n+1} = Sq^{4n+2} Sq^1$

$Sq^3 Sq^{4n+2} = 0$

$Sq^{2n-1} Sq^n = 0$

Chapter 4

Application: Hopf Invariant One Problem

In this chapter, we present an application of the Steenrod squares.

4.1 The Problem

To begin with, let us first set up the problem.

Assume $n \geq 2$ for this section. Take an oriented $2n$-cell and consider its boundary as S^{2n-1}. Then we can map $\partial e^{2n} \simeq S^{2n-1}$ into an oriented n-sphere S^n through some map f. Let us attach e^{2n} to S^n under this map f. In other words, we form the cell complex $S^n \cup_f e^{2n}$ by first forming the disjoint union $S^n \amalg e^{2n}$ and then identifying each point in ∂e^{2n} with its image under f. We clearly have the integral cohomology

$$H^m(S^n \cup_f e^{2n}) = \begin{cases} \mathbb{Z} & m = 0, n, 2n; \\ 0 & \text{otherwise.} \end{cases}$$

Denote the generator of the cohomology group in dimension n by σ, and that in dimension $2n$ by τ. The elements σ and τ agree with the respective orientations. Then $\sigma^2 = \sigma \smile \sigma$ is an integral multiple of τ.

Definition 4.1.1. The *Hopf invariant* of f is the integer $H(f)$ such that $\sigma^2 = H(f) \cdot \tau$.

Note that the homotopy type of $S^n \cup_f e^{2n}$ is determined by the homotopy class of f, and thus $H(f)$ is also determined by the homotopy class of f. Therefore, instead of saying the Hopf invariant of some map f, we can say the Hopf invariant of some homotopy class $[f]$. In this way, we have defined a transformation $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$.

The Hopf invariant one problem asks: *for which positive integers n do there exist maps of Hopf invariant one?*

Recall that the cup product is graded commutative, hence if n is odd, we have

$$\sigma^2 = \sigma \smile \sigma = (-1)^{n^2}(\sigma \smile \sigma) = -\sigma^2.$$

This implies $\sigma^2 = 0$ and thus $H(f) = 0$ for all possible f. Therefore, our options are then restricted to even integers.

Note that if n is 2, 4, or 8, then there does exist a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant one. These are known as the *Hopf maps*. For example, when $n = 2$, we have the familiar Hopf map $h\colon S^3 \to S^2$.

4.2 A Partial Solution

Before we present the partial solution to the Hopf invariant one problem obtained using the Steenrod squares, we first state the full solution. An in-depth proof of the result can be found in [Ada60], but we will not discuss it. The machinery needed to understand the proof goes way beyond this thesis.

Theorem 4.2.1 (Non-Existence of Elements of Hopf Invariant One)**.** *Unless* $n = 1, 2, 4, 8$*, there is no map* $f\colon S^{2n-1} \to S^n$ *of Hopf invariant one.*

Proof. See [Ada60]. □

We will prove the following partial solution at the end of this section.

Theorem 4.2.2. *Unless* n *is a power of 2, there is no map* $f\colon S^{2n-1} \to S^n$ *of Hopf invariant one.*

Before we prove it, we need to first make some definitions.

Definition 4.2.3. We say Sq^i is *decomposable* if $Sq^i = \sum_{t<i} a_t Sq^t$, where each a_t is a sequence of squaring operations. We say Sq^i is *indecomposable* if no such relation exists.

Example 4.2.4. Sq^1 is indecomposable. Sq^2 is indecomposable, since the only way it can possibly be decomposed is Sq^1Sq^1, but we know $Sq^1Sq^1 = 0$. Sq^3 is decomposable, since $Sq^3 = Sq^1Sq^2$ by the Adem relations. Sq^6 is decomposable, since $Sq^2Sq^4 = Sq^6 + Sq^5Sq^1$ by the Adem relations.

We will now prove a condition that determines whether a given Sq^i is decomposable or not. The following lemma will be useful.

Lemma 4.2.5. *Let p be a prime, and let a, b have the p-adic expansions $a = \sum_{i=0}^{m} a_i p^i$, $b = \sum_{i=0}^{m} b_i p^i$, where $0 \leq a_i, b_i < p$ for all i. Then*

$$\binom{b}{a} \equiv \prod_{i=0}^{m} \binom{b_i}{a_i} \quad (mod\ p).$$

Proof. Recall that in the polynomial ring $\mathbb{Z}/p[x]$, we have $(1+x)^p = 1 + x^p$. Hence, $(1+x)^b = (1+x)^{\sum b_i p^i} = \prod (1+x)^{b_i p^i} \equiv \prod (1 + x^{p^i})^{b_i}$.

By doing a binomial expansion on $(1+x)^b$, we see that $\binom{b}{a}$ is the coefficient of x^a in this expansion. Similarly, by doing a binomial expansion on $(1+x^{p^i})^{b_i}$, we see that $\binom{b_i}{a_i}$ is the coefficient of x^{a_i} in this expansion. Using the p-adic expansion of a, we then have that $\prod \binom{b_i}{a_i}$ is exactly the coefficient of x^a (mod p) in $\prod (1+x^{p^i})^{b_i}$. $\square$

With the help of the above lemma, we will prove the following theorem.

Theorem 4.2.6. *Sq^i is indecomposable if and only if i is a power of 2.*

Proof. Suppose i is a power of 2. Recall that $H^*(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$ is $\mathbb{Z}/2$ in every dimension n (an ordinary exercise in the universal coefficient theorem). Let α be the generator of $H^1(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$, then we have $Sq(\alpha^i) = (Sq\alpha)^i = (\alpha + \alpha^2)^i \equiv \alpha^i + \alpha^{2i}$ (mod 2). Hence, $Sq^t(\alpha^i) = 0$ unless $t = 0$ or i. $Sq^0(\alpha^i) = \alpha^i$ and $Sq^i(\alpha^i) = \alpha^{2i}$. If

Sq^i is decomposable, then $\alpha^{2i} = Sq^i(\alpha^i) = \sum_{t<i} a_t Sq^t(\alpha^i) = 0$, contradicting $\alpha^{2i} \neq 0$. Therefore, Sq^i is indecomposable.

Conversely, suppose $i = a + 2^k$ for some $0 < a < 2^k$. Let $b = 2^k$, then we have by the Adem relations

$$Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum_{c>0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$

Since b is a power of 2, $b-1$ is the sum of some continue powers of 2. Then by the above lemma (also recall the Pascal's triangle mod 2), we have $\binom{b-1}{a} \equiv 1 \pmod 2$. By the Adem relation above, this implies $Sq^i = Sq^{a+b}$ is decomposable. $\square$

Finally, let us prove the partial solution to the Hopf invariant one problem, Theorem 4.2.2.

Proof. By definition, $f\colon S^{2n-1} \to S^n$ has an odd Hopf invariant if and only if $\sigma^2 = \tau$ in the mod 2 cohomology, where σ and τ are the generators of the cohomology groups in dimensions n and $2n$, respectively. But σ lives in degree n, thus $\sigma^2 = Sq^n(\sigma)$. If n is not a power of 2, by Theorem 4.2.6 Sq^n is decomposable. However, $Sq^n(\sigma) = \sigma^2 = \tau$ is non-zero, whereas $Sq^i(\sigma)$ must be zero for all $0 < i < n$ for dimensional reasons. Thus Sq^n cannot be decomposable, and we have a contradiction.

Therefore, if n is not a power of 2, then f cannot have an odd Hopf invariant. Contrapositively, if f has Hopf invariant one, then n must be a power of 2. $\square$

Chapter 5

The Steenrod Algebra

In this chapter, we study the algebraic structure of the Steenrod algebra $\mathcal{A}$, which is a graded Hopf algebra over $\mathbb{Z}/2$ generated by the Steenrod squares subject to the Adem relations. We also briefly study its dual. In addition, we give a set of indecomposable generators for $\mathcal{A}$ as a $\mathbb{Z}/2$-algebra, as well as two different bases for $\mathcal{A}$ as a $\mathbb{Z}/2$-module. We assume the basic knowledge of Hopf algebras.

5.1 The Steenrod Algebra $\mathcal{A}$

First, let us recall some basic definitions.

Definition 5.1.1. Let R be a unital commutative ring and M an R-module. We define the *tensor algebra* $\Gamma(M)$ as the following. Let $M^0 = R$, $M^1 = M$, $M^2 = M \otimes M$, and in general $M^n = M^{\otimes n}$. Then $\Gamma(M)$ is the graded R-algebra defined by $\Gamma(M)_k = M^k$, where the multiplication is given by the canonical isomorphism $M^s \otimes M^t \cong M^{s+t}$.

Remark 5.1.2. Clearly, tensor algebras are free and associative but not in general commutative.

Definition 5.1.3. Let A be an algebra over a unital ring R. A *left ideal* of A is a subalgebra $I \subseteq A$ such that $ax \in I$ whenever $a \in A$ and $x \in I$. Similarly, a *right ideal* of A is a subalgebra $I \subseteq A$ such that $xa \in I$ whenever $a \in A$ and $x \in I$. A *two-sided ideal* of A is a subalgebra which is both a left ideal and a right ideal. These notions coincide when A is commutative.

Now let M be the graded $\mathbb{Z}/2$-module with $M_i = \mathbb{Z}/2$ generated by the symbol Sq^i for every $i \geq 0$. We say Sq^i has degree i. Note that M is in fact a $\mathbb{F}_2$-vector space. Form the tensor algebra $\Gamma(M)$ over M. For each pair of integers (a, b) with $0 < a < 2b$, let

$$R_{a,b} = Sq^a Sq^b + \sum_{c=0}^{\lfloor a/2 \rfloor} \begin{pmatrix} b-c-1 \\ a-2c \end{pmatrix} Sq^{a+b-c} Sq^c,$$

where $Sq^a Sq^b$ is understood to be the multiplication in $\Gamma(M)$. Let R denote the two-sided ideal of $\Gamma(M)$ generated by all such $R_{a,b}$ and $1 + Sq^0$. In other words, R is the two-sided ideal generated by the Adem relations.

Definition 5.1.4. The *Steenrod Algebra* $\mathcal{A}$ is the quotient algebra $\Gamma(M)/R$.

Note that $\mathcal{A}$ does not inherit a grading from the gradation on $\Gamma(M)$ because of the Adem relations (i.e. because $\mathcal{A}$ is no longer free), but the Adem relations let $\mathcal{A}$ inherit a grading from M, which is graded by the degrees of the symbols Sq^i ($\deg Sq^i = i$). Elements in $\mathcal{A}$ are polynomials in Sq^i ($i \geq 0$) with coefficients in $\mathbb{Z}/2$ and subject to the Adem relations.

Once we know what $\mathcal{A}$ is algebraically, let us find a set of generators that generate $\mathcal{A}$ as an algebra.

Definition 5.1.5. Let R be a unital commutative ring and A a graded R-algebra. We can consider R as a graded R-algebra by the convention $R_0 = R$ and $R_i = 0$ for all $i \neq 0$. If we are given an algebra homomorphism $\varepsilon\colon A \to R$, then we say A is *augmented*. An augmented R-algebra is *connected* if $\varepsilon_0\colon A_0 \to R$ is an isomorphism.

Let A be a connected graded R-algebra, and let $\bar{A}$ denote the kernel of the augmentation ε. Then by the above definition, $\bar{A}$ is the ideal containing all the elements of positive degree.

Definition 5.1.6. $\bar{A}$ is called the *augmentation ideal*.

Definition 5.1.7. The ideal of *decomposable* elements of A is the ideal $\mu(\bar{A} \otimes \bar{A}) \subseteq A$, where μ is the multiplication in A.

Note that the definition of decomposable elements given above is consistent with Definition 4.2.3. Recall in Chapter 4, we proved that Sq^i is decomposable if and only if i is not a power of 2. As a consequence, we have the following theorem.

Theorem 5.1.8. $\{Sq^{2^i}\}_{i\geq 0}$ *generate* $\mathcal{A}$ *as an algebra.*

Proof. $\{Sq^i\}_{i\geq 0}$ clearly generate $\mathcal{A}$ as an algebra, thus $\{Sq^i \mid i \geq 0,\ Sq^i \text{ indecomposable}\}$ also generate $\mathcal{A}$ as an algebra, but this set is exactly $\{Sq^{2^i}\}_{i\geq 0}$ by Theorem 4.2.6. □

In fact, $\mathcal{A}$ is not just a graded $\mathbb{Z}/2$-algebra, it also possesses an additional structure of a Hopf algebra. Our next goal in this section is to figure out the comultiplication in $\mathcal{A}$. As a reminder, recall that a *Hopf algebra* H is a bialgebra (i.e. an algebra that is also a coalgebra, and these two structures are compatible) with an *antipode*, which can be thought of as the convolutional inverse to the identity map on H. We refer to any introductory text on Hopf algebras for these concepts, for example [Rad12].

Before we proceed, we make some definitions that will come up frequently in our later discussions.

Definition 5.1.9. Given a sequence $I = \{i_1, \ldots, i_r\}$ of positive integers, we denote by Sq^I the composition $Sq^{i_1} \ldots Sq^{i_r}$. By convention, if I is empty, then $Sq^I = Sq^0$.

Definition 5.1.10. The *length* of any sequence I is the number of terms in the sequence. The *degree* $d(I)$ of any sequence I is the sum of the terms, i.e. $d(I) = \sum_j i_j$.

Definition 5.1.11. A sequence I is *admissible* if $i_j \geq 2(i_{j+1})$ for every $j < r$.

Definition 5.1.12. For an admissible sequence I, the *excess* $e(I)$ is $2i_1 - d(I)$.

Note that we can rewrite the excess in the following way:

$$\begin{aligned} e(I) &= 2i_1 - d(I) \\ &= i_1 - i_2 - \cdots - i_r \\ &= (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_r). \end{aligned}$$

The last expression justifies the name, because it shows $e(I)$ is measuring how far away I is from not being admissible, but for computational convenience we will use either of the first two expressions.

Now we come back to the comultiplication in $\mathcal{A}$. Let $\Gamma(M)$ be the same tensor algebra as before, with $M_i = \mathbb{Z}/2 = \langle Sq^i \rangle$. Define the comultiplication on $\Gamma(M)$ by

$$\begin{aligned} \Delta\colon \Gamma(M) &\longrightarrow \Gamma(M) \otimes \Gamma(M) \\ Sq^i &\longmapsto \sum_j Sq^j \otimes Sq^{i-j}. \end{aligned}$$

By the definition of a Hopf algebra, we must require that Δ is an algebra homomorphism. That is,

$$\Delta(Sq^r \otimes Sq^s) = \Delta(Sq^r) \otimes \Delta(Sq^s) = \left(\sum_a Sq^a \otimes Sq^{r-a} \right) \otimes \left(\sum_b Sq^b \otimes Sq^{s-b} \right).$$

Theorem 5.1.13. Δ *defined above induces an algebra homomorphism* $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Proof sketch. Since $\mathcal{A}$ is a quotient of $\Gamma(M)$, let $p\colon \Gamma(M) \to \mathcal{A}$ be the natural projection. It suffices to show $\ker p \subseteq \ker \Delta$. Let K_n be the same as in Chapter 3, i.e. the topological product of n copies of $K(\mathbb{Z}/2, 1)$. Define the following map

$$w\colon \mathcal{A} \longrightarrow H^*(K_n; \mathbb{Z}/2)$$

$$\theta \longmapsto \theta(\sigma_n),$$

where σ_n is the elementary symmetric function of degree n. In other words, w is evaluation on σ_n. Note that w raises degree by n. It turns out that, by Lemma 5.1.14 and Theorem 5.1.16, w is a monomorphism through degree n. Similarly, we define w' as evaluation on σ_{2n}. Consider the diagram

$$\begin{array}{ccccccc}
\Gamma(M) & \xrightarrow{\quad p \quad} & \mathcal{A} & & & \xrightarrow{\qquad w' \qquad} & \\
{\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle w\times w} & & & & \\
\mathcal{A}\otimes\mathcal{A} & \xrightarrow{w\otimes w} & H^*(K_n)\otimes H^*(K_n) & \xrightarrow[\sim]{\kappa} & H^*(K_n\times K_n) & = & H^*(K_{2n})
\end{array}$$

where cohomology is over $\mathbb{Z}/2$ coefficients. Note that the isomorphism κ comes from the Künneth theorem. We will show the outer part of the above diagram is commutative. Observe,

$$\begin{aligned}
w'p(Sq^i) = w'(Sq^i) &= Sq^i(\sigma_{2n}) \\
&= Sq^i(\sigma_n \times \sigma_n) \\
&= \sum_j Sq^j(\sigma_n) \times Sq^{i-j}(\sigma_n) \\
&= (w \otimes w)(\Delta(Sq^i)),
\end{aligned}$$

following from the definitions of the various maps and the Cartan formula. This shows that the outer part of the above diagram commutes on Sq^i.

Note that a basis for $M^{\otimes r}$ is given by $\{Sq^{i_1} \otimes \cdots \otimes Sq^{i_r}\}$. Let us denote $Sq^{i_1} \otimes \cdots \otimes Sq^{i_r}$ by $Sq^I_{\otimes}$, where the sequence $I = \{i_1, \ldots, i_r\}$ need not be admissible. Similar to the above computation, we have

$$\begin{aligned} w'p(Sq^I_{\otimes}) = w'(Sq^I) &= Sq^I(\sigma_{2n}) \\ &= Sq^{i_1} \dots Sq^{i_{r-1}} \sum_s Sq^s(\sigma_n) \times Sq^{i_r - s}(\sigma_n) \\ &= \sum_{I_1 + I_2 = I} Sq^{I_1}(\sigma_n) \times Sq^{I_2}(\sigma_n) \\ &= \dots \\ &= \kappa(w \otimes w)(\Delta(Sq^I_{\otimes})), \end{aligned}$$

where $I_1 + I_2$ is interpreted as entry-wise addition. Since this holds for a basis element, it holds for all elements. Hence, the outer part of the above diagram is indeed commutative.

To finish the proof, suppose $p(z) = 0$ for some $z \in \Gamma(M)$. Then by the commutativity we just showed, $\kappa(w \otimes w)(\Delta(z)) = w'p(z) = 0$. However, κ is an isomorphism, and we can choose n large enough so that $w \otimes w$ is a monomorphism on elements of degree $\deg z$. Therefore, $\Delta(z) = 0$. □

Lemma 5.1.14. *If $d(I) \leq n$, then $Sq^I(\sigma_n)$ can be expressed as $\sigma_n \cdot Q_I$, where $Q_I = \sigma_{i_1} \dots \sigma_{i_r} + \sum$ (monomials of lower order).*

Proof. A routine verification using the Cartan formula and induction. See [MT68, Chapter 3]. □

Remark 5.1.15. Note that the multiplication in $\mathcal{A}$ is associative (inherited from the associativity of $\Gamma(M)$) but not commutative (as an easy counter-example, $Sq^1 Sq^2 =$

$Sq^3 \neq Sq^2 Sq^1$). However, the comultiplication in $\mathcal{A}$ is both coassociative and cocommutative. We omit the proof, since it is a routine verification on the generators.

So far, we have studied the (Hopf) algebra structure of $\mathcal{A}$, and in Theorem 5.1.8 we gave a set of indecomposable generators for $\mathcal{A}$ as a $\mathbb{Z}/2$-algebra. In this last part of this section, we give a basis for $\mathcal{A}$ as a $\mathbb{Z}/2$-module.

Theorem 5.1.16 (Serre-Cartan Basis). *The monomials Sq^I, as I runs through all admissible sequences, form a basis for $\mathcal{A}$ as a $\mathbb{Z}/2$-module.*

Proof sketch. Linearly independent: This can either be directly deduced from Serre's theorem (Theorem 6.3.5), or deduced by the following argument. Let S denote the ring of symmetric polynomials $\mathbb{Z}/2[\sigma_1, \ldots, \sigma_n]$, then as I traverses over all admissible sequences of degree $\leq n$, the monomials $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_r}$ are linearly independent in S. But note that S is a subring of $H^*(K_n; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, \ldots, x_n]$, where K_n is again the same as in Chapter 3, thus these monomials are also linearly independent in $H^*(K_n; \mathbb{Z}/2)$. It follows from Lemma 5.1.14 that as I runs over all admissible sequences of degree $\leq n$, $Sq^I(\sigma_n)$ are linearly independent. Take a map $f\colon K_n \to K(\mathbb{Z}/2, n)$ such that the induced map f^* takes ι_n to σ_n, it then follows that $Sq^I(\iota_n)$ are linearly independent in $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$. Therefore, Sq^I are linearly independent for I admissible.

Span: We need an *ad hoc* definition. For any sequence $I = \{i_1, \ldots, i_r\}$, we define the *moment* of I to be $m(I) = \sum_s i_s s$. Now take a non-admissible sequence I and consider Sq^I. Since I is not admissible, there is some pair i_s, i_{s+1} with $i_s < 2i_{s+1}$. Take the right-most such pair and apply the Adem relation, we obtain a sum of monomials with moment strictly smaller than I. We continue doing this procedure for the newly obtained sequences. Since the moment function is bounded below, the procedure eventually terminates and we have expressed the non-admissible sequence I as a sum of admissible sequences. Hence, $\{Sq^I\}$ with I admissible span $\mathcal{A}$. □

Example 5.1.17. $\mathcal{A}_7$, as a $\mathbb{Z}/2$-module, has the Serre-Cartan basis

$$\{Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1\}.$$

5.2 The Dual of the Steenrod Algebra $\mathcal{A}^*$

In this section, we briefly study the dual of the Steenrod algebra. The purpose of this section is mainly to prepare for the discussion on the Milnor basis.

Let us first make clear what we mean by a dual Hopf algebra. Let k be a field and A a (connected) Hopf algebra over k. Moreover, we assume A is of finite type for simplicity. In other words, we assume A_i is finitely generated over k for all i. Under this setting, we make the following definition.

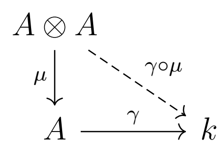
Definition 5.2.1. The *dual Hopf algebra* to A, which we denote A^*, is defined by $(A^*)_i = (A_i)^*$. That is, the degree-i component of A^* is dual to the degree-i component of A as a vector space over k. The multiplication in A induces the comultiplication in A^*, and the comultiplication in A induces the multiplication in A^*.

Specifically, the multiplication $\mu\colon A \otimes A \to A$ in A induces the comultiplication in A^* in the following way. First, we observe that we can identify $(A \otimes A)^*$ with $A^* \otimes A^*$ by the isomorphism

$$A^* \otimes A^* \longrightarrow (A \otimes A)^*$$

$$\alpha \otimes \beta \longmapsto [x \otimes y \mapsto \alpha(x)\beta(y)],$$

where $\alpha, \beta \in A^*$ and $x, y \in A$. Now, suppose we have a linear functional $\gamma\colon A \to k$, i.e. $\gamma \in A^*$. This leads to a linear functional $(A \otimes A)^* \to k$ through the following commutative diagram



Therefore, the composition $- \circ \mu$ induces a map $A^* \to (A \otimes A)^*$. Further composing it with the inverse of the above isomorphism gives a map $A^* \to A^* \otimes A^*$, which is exactly the comultiplication in A^* we want. In a similar way, composition with the comultiplication map in A leads to the multiplication map in A^*.

Remark 5.2.2. A^* defined as above is a Hopf algebra. Moreover, the multiplication in A^* is associative (*resp.* commutative) if and only if the comultiplication in A is coassociative (*resp.* cocommutative). Similarly, the comultiplication in A^* is coassociative (*resp.* cocommutative) if and only if the multiplication in A is associative (*resp.* commutative).

Recall, at the end of Section 5.1, we remarked that $\mathcal{A}$ is associative, non-commutative, coassociative, and cocommutative. Therefore, by the above fact, $\mathcal{A}^*$ is associative, commutative, coassociative, and non-cocommutative. In the rest of this section, we will find out the generators for $\mathcal{A}^*$ as an algebra, and give another basis for $\mathcal{A}$ as a $\mathbb{Z}/2$-module.

We start by making some definitions. Let $\mathfrak{R}$ be the set

$$\mathfrak{R} = \left\{ \{i_r\}_{r \in \mathbb{N}^*} \mid i_r \in \mathbb{N}\ \forall r \text{ and } \#(i_r \neq 0) < \infty \right\}.$$

In words, $\mathfrak{R}$ consists of all infinite sequences $I = \{i_1, i_2, i_3, \ldots\}$ of non-negative integers with only finitely many non-zero entries.

Definition 5.2.3. A sequence $I \in \mathfrak{R}$ is said to be *admissible* if its head is admissible in our previously defined sense, followed by all zeros. Precisely, $I \in \mathfrak{R}$ is admissible if it is of the following form

$$I = \left\{ \underbrace{i_1, i_2, \ldots, i_r}_{\text{admissible as Def. 5.1.11}}, 0, 0, \ldots \right\}.$$

Definition 5.2.4. We denote $\mathfrak{J} := \{I \in \mathfrak{R} \mid I \text{ admissible}\}$. That is, $\mathfrak{J}$ is the subset of $\mathfrak{R}$ consisting of all admissible sequences in $\mathfrak{R}$.

Next, we define a special family of admissible sequences for notational simplicity.

Definition 5.2.5. For each integer $k \geq 0$, we let I^k be the admissible sequence $\left\{2^{k-1}, \ldots, 2, 1, 0, \ldots\right\}$. We let I^0 denote the zero sequence.

An important observation: let x be the generator of $H^1(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$, then $Sq^I(x) = x^{2^k}$ if $I = I^k$, and $Sq^I(x) = 0$ for any other admissible sequence. We can verify this fact in the following way. Given an admissible sequence, call the right-most non-zero entry the first entry, the second right-most non-zero entry the second entry, and so on. If an admissible sequence I is not of the form I^k for some $k \geq 0$, then there must be some n^{th} entry whose value is greater than 2^{n-1} (it cannot be smaller than 2^{n-1}, otherwise I will not be admissible), then it follows from property 2 of Theorem 3.2.1 that $Sq^I(x) = 0$. The fact that $Sq^{I^k}(x) = x^{2^k}$ is trivial to verify. As a consequence, we have the more general observation: for any non-zero sequence I (admissible or not), $Sq^I(x) = 0$ unless I is obtained from some I^k by inserting (a finite number of) zeros.

Definition 5.2.6. For each $i \geq 0$, let ξ_i be the element of $\mathcal{A}^*_{2^i-1}$ characterized by $\xi_i(\theta)(x^{2^i}) = \theta(x) \in H^{2^i}(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$ for all $\theta \in \mathcal{A}_{2^i-1}$. ξ_0 is the unit of $\mathcal{A}^*$. In addition, we will adopt the conventional notation $\langle \xi_i, \theta \rangle$ for $\xi_i(\theta)$.

Proposition 5.2.7. *Let I be an admissible sequence. For $k \geq 1$, $\left\langle \xi_k, Sq^I \right\rangle = 1$ if $I = I^k$. Otherwise, $\left\langle \xi_k, Sq^I \right\rangle = 0$. Further, for an arbitrary sequence I, $\left\langle \xi_k, Sq^I \right\rangle = 0$ unless I is obtained from I^k by inserting (finitely many) zeros.*

Proof. This follows immediately from our observation above. □

We define a map $\gamma \colon \mathfrak{J} \to \mathfrak{R}$ by

$$\gamma \colon \{i_1, \ldots, i_k, 0, \ldots\} \longmapsto \{i_1 - 2i_2, i_2 - 2i_3, \ldots, i_k, 0, \ldots\}.$$

Observe that although $\mathfrak{J}$ is a proper subset of $\mathfrak{R}$, γ is in fact a bijection. Indeed, we can explicitly give the inverse of γ:

$$\gamma^{-1}\colon \{r_1, r_2, \ldots, r_k, 0, \ldots\} \longmapsto \left\{\sum_{i=1}^{k} 2^{i-1} r_i, \sum_{i=2}^{k} 2^{i-2} r_i, \sum_{i=3}^{k} 2^{i-3} r_i, \ldots, r_k, 0, \ldots\right\}.$$

Moreover, for each $R = \{r_1, r_2, \ldots\} \in \mathfrak{R}$, we define $\xi^R \in \mathcal{A}^*$ by $\xi^R = \prod_{i=1}^{\infty} (\xi_i)^{r_i}$. By the definition of γ, we observe that for some $I \in \mathfrak{J}$, the degree of Sq^I is the same as the degree of $\xi^{\gamma(I)}$. We give a total order to the sequences in $\mathfrak{J}$ lexicographically from the right. For example,

$$\{11, 5, 2, 0, \ldots\} > \{8, 3, 1, 0, \ldots\} > \{9, 4, 0, \ldots\} > \{8, 2, 0, \ldots\}.$$

Theorem 5.2.8. *For two admissible sequences $I, J \in \mathfrak{J}$ (assuming $J \geq I$), $\langle \xi^{\gamma(J)}, Sq^I \rangle = 0$ if $I < J$, $\langle \xi^{\gamma(J)}, Sq^I \rangle = 1$ if $I = J$.*

Proof sketch. The proof proceeds by a downward induction. Let $J = \{a_1, \ldots, a_k, 0, \ldots\}$ and $I = \{b_1, \ldots, b_k, 0, \ldots\}$. Since we assume $J \geq I$, there is $a_k \geq b_k$. Let

$$J' = \left\{a_1 - 2^{k-1}, a_2 - 2^{k-2}, \ldots, a_k - 1, 0, \ldots\right\}.$$

By our definition above,

$$\xi^{\gamma(J)} = \prod_{i=1}^{k-1} (\xi_i)^{a_i - 2a_{i+1}} \cdot \xi_k^{a_k} = \left(\prod_{i=1}^{k-1} (\xi_i)^{a_i - 2a_{i+1}} \cdot \xi_k^{a_k - 1}\right) \cdot \xi_k = \xi^{\gamma(J')} \xi_k.$$

Therefore, we have

$$\begin{aligned}
\langle \xi^{\gamma(J)}, Sq^I \rangle &= \langle \xi^{\gamma(J)} \xi_k, Sq^I \rangle \\
&= \left\langle \Delta^*(\xi^{\gamma(J')} \otimes \xi_k), Sq^I \right\rangle \\
&= \left\langle \xi^{\gamma(J')} \otimes \xi_k, \Delta(Sq^I) \right\rangle,
\end{aligned}$$

where Δ is the comultiplication in $\mathcal{A}$, and Δ^* is the multiplication in $\mathcal{A}^*$ that Δ induces. Recall we explicitly derived Δ in Section 5.1, hence

$$\begin{aligned}\left\langle \xi^{\gamma(J)}, Sq^I \right\rangle &= \left\langle \xi^{\gamma(J')} \otimes \xi_k, \sum Sq^{I_1} \otimes Sq^{I_2} \right\rangle \\ &= \sum \left\langle \xi^{\gamma(J')}, Sq^{I_1} \right\rangle \left\langle \xi_k, Sq^{I_2} \right\rangle\end{aligned}$$

by linearity, where the sum is over sequences I_1, I_2 such that $I_1 + I_2 = I$.

If $b_k = 0$, the k^{th} entry, and all entries after it, of I_2 are 0. It follows from Proposition 5.2.7 that $\left\langle \xi_k, Sq^{I_2} \right\rangle = 0$. If $b_k \neq 0$, then the only non-zero term in the above sum is at $I_2 = I^k$. Consequently, $\left\langle \xi^{\gamma(J)}, Sq^I \right\rangle = \left\langle \xi^{\gamma(J')}, Sq^{I-I^k} \right\rangle$. Inductively reducing k yields the desired result. $\square$

Corollary 5.2.9. *As an algebra, $\mathcal{A}^*$ is the polynomial ring over $\mathbb{Z}/2$ generated by $\{\xi_i\}_{i\geq 1}$.*

Proof sketch. Notice that the statement of the above theorem is identical to the definition of a dual basis in linear algebra, thus the above theorem shows $\left\{\xi^{\gamma(J)} \mid J \in \mathfrak{J}\right\}$ form a basis for $\mathcal{A}^*$ (as a vector space). But since γ is a bijection between $\mathfrak{J}$ and $\mathfrak{R}$, as J runs through $\mathfrak{J}$, $\xi^{\gamma(J)}$ runs through all the monomials in ξ_i. However, a polynomial ring is exactly a ring where the monomials in the generators form a basis as a vector space. Therefore, $\mathcal{A}^*$ is a $\mathbb{Z}/2$-polynomial ring with generators $\{\xi_i\}_{i\geq 1}$. $\square$

We noted above that $\left\{\xi^{\gamma(J)} \mid J \in \mathfrak{J}\right\}$ form a vector-space basis for $\mathcal{A}^*$. Since γ is a bijection, this implies $\left\{\xi^R \mid R \in \mathfrak{R}\right\}$ form a basis for $\mathcal{A}^*$. From this, we can define another basis for $\mathcal{A}$ as a module.

Definition 5.2.10. The dual basis of $\left\{\xi^R \mid R \in \mathfrak{R}\right\}$, whose elements we denote $\left\{Sq^R \mid R \in \mathfrak{R}\right\}$, is called the *Milnor basis* for $\mathcal{A}$ as a $\mathbb{Z}/2$-module.

Remark 5.2.11. By definition of the dual basis, $\left\langle \xi^R, Sq^{R'} \right\rangle = 1$ if $R = R'$, and $\left\langle \xi^R, Sq^{R'} \right\rangle = 0$ if $R \neq R'$.

We note that the Milnor basis is completely different from the Serre-Cartan basis we introduced in Section 5.1, albeit they have similar notations. For example, the element $Sq^{\{2,1,0,\ldots\}}$ in the Milnor basis is different from the element $Sq^2 Sq^1$ in the Serre-Cartan basis, although we tend to use the abbreviated notation $Sq^{2,1}$ for both of them. However, we do have the coincidence $Sq^{\{i,0,\ldots\}} = Sq^i$, see [MT68, Chapter 6].

Chapter 6

Cohomology of Eilenberg-MacLane Spaces

The computational part of this thesis starts from this chapter. In this chapter, we try to compute the cohomology of two Eilenberg-MacLane spaces. Specifically, we compute the integral cohomology of $K(\mathbb{Z}, 2)$ and the mod-2 cohomology of $K(\mathbb{Z}/2, 2)$. These computations not only serve as a demonstration of the Serre spectral sequence, but also lead to concepts and results that will become important in our later computations. In particular, we obtain the ring structures of $H^*(K(\mathbb{Z}/2, q); \mathbb{Z}/2)$ and $H^*(K(\mathbb{Z}, q); \mathbb{Z}/2)$ for all positive integers q. We assume the basic knowledge of spectral sequences.

6.1 Fibrations

We start with some basic definitions.

Definition 6.1.1. A map $p\colon E \to B$ is said to have the *homotopy lifting property* with respect to a space X if, given a homotopy $G\colon X \times I \to B$ and a map $g\colon X \to E$ lifting $G|_{X\times\{0\}}$, i.e. $G|_{X\times\{0\}} = p \circ g$, there exists a (not necessarily unique) homotopy $\tilde{G}\colon X \times I \to E$ lifting G with $g = \tilde{G}|_{X\times\{0\}}$.

The above condition is depicted by the following commutative diagram.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{g} & E \\ {\scriptstyle i}\big\downarrow & \overset{\tilde{G}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[G]{} & B \end{array}$$

Definition 6.1.2. A *Hurewicz fibration* is a map $p\colon E \to B$ satisfying the homotopy lifting property for all spaces X.

Definition 6.1.3. A *Serre fibration* is a map $p\colon E \to B$ satisfying the homotopy lifting property for all CW-complexes X.

Definition 6.1.4. Given a fibration $p\colon E \to B$, the space B is called *base space* and the space E is called *total space*. The *fiber over* $b \in B$ is the subspace $F_b = p^{-1}(b) \subseteq E$. Note that for any two points b and b' in B, F_b and $F_{b'}$ are in fact homeomorphic. Therefore, given a base point $b_0 \in B$, we call F_{b_0} the *fiber space*.

In this thesis, when we say a fibration, we always mean a Serre fibration. We denote a fibration by $F \to E \xrightarrow{p} B$.

A particularly useful type of fibration for our computations is the loop-path fibration. Given a pointed space (X, x_0), the *loop space* ΩX over X is the space of pointed maps $\mathrm{Map}_*(S^1, X)$, and the *path space* PX over X is the space of pointed maps $\mathrm{Map}_*(I, X)$. Of course, ΩX and PX are equipped with the compact-open topology. Note that the path space PX is contractible.

Definition 6.1.5. Given a pointed space X, the *loop-path fibration* is the fibration $\Omega X \to PX \xrightarrow{ev_1} X$, where ev_1 evaluates a path at the non-base-point end.

The loop-path fibration plays out nicely in the Serre spectral sequence (which will be introduced in the next section). Assuming $\pi_1(X)$ acts trivially on higher homotopy groups, we have the (cohomology) Serre spectral sequence

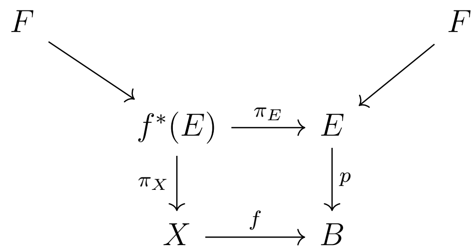
$$E_2^{p,q} = H^p(X; H^q(\Omega X)) \Longrightarrow H^{p+q}(PX),$$

but PX is contractible, thus $H^{p+q}(PX) = 0$ unless $p = q = 0$. Therefore, everything in the Serre spectral sequence must die except for the copy of coefficients at the $(0, 0)$ spot.

Another useful type of fibration is the pullback fibration (sometimes also called the induced fibration), see [Hat02, Chapter 4] or [MT68, Chapter 11]. Given a fibration $F \to E \xrightarrow{p} B$ and a map $f\colon X \to B$ for some space X, we define the pullback space $f^*(E)$ to be

$$f^*(E) = \{(x, e) \in X \times E \mid f(x) = p(e)\}.$$

Let $\pi_X \colon f^*(E) \to X$ and $\pi_E \colon f^*(E) \to E$ be the projections of $f^*(E)$ to X and E, respectively. Then the *pullback fibration* $F \to f^*(E) \xrightarrow{\pi_X} X$ fits into the following commutative diagram.



Finally, recall that given a pointed space (X, x_0), the *reduced suspension* ΣX of X is the quotient space

$$\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I).$$

In the category of pointed spaces, the reduced suspension functor is left adjoint to the loop space functor. That is, we have a natural isomorphism

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y),$$

where X and Y are pointed spaces, and Map_* means continuous maps that preserve base points.

Using this adjunction, we have that for any pointed space X,

$$\pi_n(\Omega X) = [S^n, \Omega X] \cong [\Sigma S^n, X] = [S^{n+1}, X] = \pi_{n+1}(X).$$

It is then obvious that for an Eilenberg-MacLane space $K(G, n)$, we have

$$\Omega K(G, n) \cong K(G, n-1).$$

6.2 Two Computational Examples

In this section, we will try to explicitly compute the cohomologies $H^*(K(\mathbb{Z},2);\mathbb{Z})$ and $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, using the loop-path fibration and the Serre spectral sequence.

The following result is due to Jean-Pierre Serre. A proof can be found in [Hat04].

Theorem 6.2.1 ((cohomology) Serre Spectral Sequence)**.** *Let $F \to E \xrightarrow{p} B$ be a fibration, suppose B and F are path-connected. The following holds*

1. The cochain complex $C^\bullet(E)$ admits a certain filtration, leading to a first-quadrant spectral sequence.

2. In the resulting spectral sequence $\{E_r, d_r\}$, the bidegree of d_r is $(r, 1-r)$.

3. $E_1^{p,q} = C^p(B) \otimes H^q(F)$.

4. $E_2^{p,q} = H^p(B; \mathscr{H}^q(F))$, where $\mathscr{H}^q(F)$ denotes the local coefficients. If B is simply connected, then $E_2^{p,q} = H^p(B; H^q(F))$.

5. The spectral sequence converges to $H^*(E)$. We often denote this convergence by $E_2^{p,q} \Rightarrow H^{p+q}(E)$.

We remark that in our computations, the base spaces are always simply connected, hence we do not need to deal with local coefficients. In addition, if the underlying coefficient group is a commutative ring (in our case, it is always either $\mathbb{Z}$ or $\mathbb{Z}/2$, so this condition applies), then the Künneth theorem implies

$$E_2^{p,q} = H^p(B; H^q(F;R)) = H^p(B;R) \otimes H^q(F;R),$$

where R is the coefficient ring.

Recall that given a fibration $F \xrightarrow{i} E \xrightarrow{p} B$, we have a long exact sequence in homotopy (see, for example, [Hat02]):

$$\cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots.$$

Although a fibration does not lead to a long exact sequence in (co)homology, we do have the following useful result due to Serre. The theorem and definition below have analogous statements in homology, but we will only be interested in the cohomology version.

Theorem 6.2.2 (Serre's Exact Sequence in Cohomology)**.** *If B is $(p-1)$-connected and F is $(q-1)$-connected, then there is an exact sequence that terminates as follows:*

$$\cdots \to H^{p+q-2}(F) \xrightarrow{\tau} H^{p+q-1}(B) \xrightarrow{p^*} H^{p+q-1}(E) \xrightarrow{i^*} H^{p+q-1}(F).$$

Proof. See [MT68, Chapter 8]. □

Notice that in the above exact sequence, we denoted the connecting homomorphism by τ. This map will be important in our computations.

Definition 6.2.3. The map $\tau = d_n \colon E_n^{0,n-1} \to E_n^{n,0}$ is called the *transgression.* We say $x \in H^{n-1}(F)$ is *transgressive* if $\tau(x)$ is defined (or equivalently, if $d_i(x) = 0$ for all $i < n$).

The following result plays a crucial role in our computations. It goes by the slogan "transgression commutes with the squares".

Proposition 6.2.4. *If x is transgressive, then so is $Sq^i(x)$. Moreover, if $y \in \tau(x)$, then $Sq^i(y) \in \tau(Sq^i(x))$.*

Proof sketch. It is shown in [Hat04] that $\tau(x)$ contains y if and only if $\delta(x) = p^*(y)$, where δ is the coboundary map. Therefore, $y \in \tau(x)$ implies $p^*(y) = \delta(x)$, and hence $Sq^i(p^*(y)) = Sq^i(\delta(x))$. It follows from naturality that $p^*(Sq^i(y)) = \delta(Sq^i(x))$, which then implies $Sq^i(y) \in \tau(Sq^i(x))$. □

6.2.1 $H^*(K(\mathbb{Z}, 2); \mathbb{Z})$

Recall that S^1 is a model of $K(\mathbb{Z}, 1)$, thus the cohomology of $K(\mathbb{Z}, 1)$ is completely known. Specifically,

$$H^n(K(\mathbb{Z}, 1); \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.2.5. *$H^*(K(\mathbb{Z}, 2); \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[\iota_2]$, where ι_2 is of dimension 2. That is, ι_2^n generates $H^{2n}(K(\mathbb{Z}, 2); \mathbb{Z})$.*

Proof. Consider the loop-path fibration

$$\begin{array}{ccc} \Omega K(\mathbb{Z}, 2) \cong K(\mathbb{Z}, 1) & \longrightarrow & PK(\mathbb{Z}, 2) \\ & & \downarrow \\ & & K(\mathbb{Z}, 2) \end{array}$$

and consider the (cohomology) Serre spectral sequence with $\mathbb{Z}$ coefficients. On the p-axis, we have the cohomology of the base space $H^*(K(\mathbb{Z}, 2); \mathbb{Z})$. On the q-axis, we have the cohomology of the fiber space $H^*(K(\mathbb{Z},1);\mathbb{Z})$. The spectral sequence converges to the cohomology of the total space $PK(\mathbb{Z},2)$. But we know the path space $PK(\mathbb{Z},2)$ is contractible, thus $E_\infty^{p,q} = 0$ unless $p = q = 0$. Therefore, we do not need to worry about the group $\mathbb{Z}$ at $(p,q) = (0,0)$ on the E_2-page, as it will persist to the E_∞-page. However, we must eventually eliminate all the other non-zero groups on the E_2-page.

Since the cohomology of the fiber space $K(\mathbb{Z},1)$ is zero above dimension 1, it follows that $E_2^{0,q} = 0$ for all $q > 1$, and thus $E_2^{p,q} = 0$ for all $p \geq 0$ and $q > 1$. As a consequence, the only differential that can be non-zero is d_2, since all higher differentials d_r $(r \geq 3)$ will either map from zero or map to zero.

So far, we have figured out $E_2^{0,0} \cong E_2^{0,1} = \mathbb{Z}$, and $E_2^{p,q} = 0$ for all $p \geq 0$ and $q > 1$. Consider the $(1,0)$ spot on the E_2-page, since the base space $K(\mathbb{Z},2)$ is simply connected, we must have $E_2^{1,0} = H^1(K(\mathbb{Z},2);\mathbb{Z}) = 0$. As a consequence, $E_2^{1,1} = 0$.

Now consider the $(2,0)$ spot on the E_2-page. This is the only spot on all pages that $E_2^{0,1} = \mathbb{Z}$ can map to, as we noted d_2 is the only possible non-zero differential. Therefore, we must have a copy of $\mathbb{Z}$ at the $(2,0)$ spot on the E_2-page to eliminate the group $\mathbb{Z}$ at the $(0,1)$ spot (i.e. to make d_2 exact at $E_2^{0,1}$). On the other hand, we cannot have more than a copy of $\mathbb{Z}$ at the $(2,0)$ spot, since otherwise d_2 will not be exact at $E_2^{2,0}$ and the non-zero quotient will persist to the E_∞-page. Hence, $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is an isomorphism, and it must take generator to generator. Denote the generator of $E_2^{0,1} = \mathbb{Z}$ by ι_1, then we let $\iota_2 \colon = d_2(\iota_1)$ generate $E_2^{2,0} = \mathbb{Z}$. By the

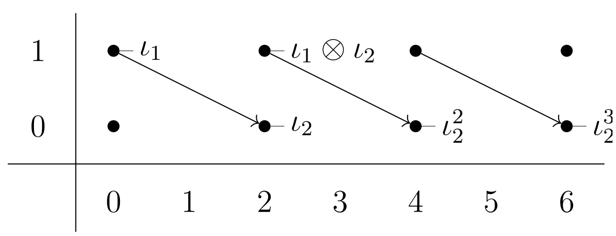
Serre spectral sequence, it then follows that $E_2^{2,1} = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ is generated by $\iota_1 \otimes \iota_2$.

Next, consider the $(3,0)$ spot on the E_2-page. Since no non-zero group maps to this spot and this spot maps to no non-zero group, we must have $E_2^{3,0} = 0$. As a consequence, $E_2^{3,1} = 0$. Next, consider the $(4,0)$ spot on the E_2-page. Similar to the $(2,0)$ spot, this is the only spot on all pages that $E_2^{2,1} = \mathbb{Z}$ can map to, and thus we must have a copy of $\mathbb{Z}$ at this spot to eliminate $E_2^{2,1}$. Also, we cannot have more than a copy of $\mathbb{Z}$ for the same reason as above. Therefore, $E_2^{4,0} = \mathbb{Z}$. Again, $d_2 \colon E_2^{2,1} \to E_2^{4,0}$ is an isomorphism, and thus $d_2(\iota_1 \otimes \iota_2)$ is the generator of $E_2^{4,0}$. But by the Leibniz rule, we have

$$
\begin{aligned}
d_2(\iota_1 \otimes \iota_2) &= d_2(\iota_1) \otimes \iota_2 + \iota_1 \otimes d_2(\iota_2) \\
&= \iota_2 \otimes \iota_2,
\end{aligned}
$$

where $d_2(\iota_2) = 0$ because d_2 maps into the fourth quadrant from $E_2^{2,0}$. Hence, ι_2^2 generates $E_2^{4,0} = H^4(K(\mathbb{Z}, 2); \mathbb{Z})$.

The above computation is illustrated in the spectral sequence figure below.



Continuing with this computation, we obtain the integral cohomology of $K(\mathbb{Z}, 2)$:

$$
H^n(K(\mathbb{Z}, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z} = \left\langle \iota_2^{n/2} \right\rangle & n \text{ even;} \\ 0 & n \text{ odd.} \end{cases}
$$

This finishes the proof. □

6.2.2 $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$

Now we turn to a much more complicated computation. The strategy does not change: we start with a loop-path fibration, consider its Serre spectral sequence, and eliminate unwanted groups.

Consider the loop-path fibration

$$\begin{array}{ccc} \Omega K(\mathbb{Z}/2,2) \cong K(\mathbb{Z}/2,1) & \longrightarrow & PK(Z/2,2) \\ & & \downarrow \\ & & K(\mathbb{Z}/2,2) \end{array}$$

and consider the (cohomology) Serre spectral sequence with $\mathbb{Z}/2$ coefficients. Again, we have $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ on the p-axis, and $H^*(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$ on the q-axis. The total space is contractible, thus $E_\infty^{p,q} = 0$ unless $p = q = 0$. Therefore, everything except for the group $\mathbb{Z}/2$ at the $(0,0)$ spot should be eventually eliminated.

Recall in Chapter 3 we mentioned that the cohomology of the fiber space $K(\mathbb{Z}/2,1)$ is the polynomial ring $\mathbb{Z}/2[\alpha]$ on one generator α of dimension 1. Since we know the cohomology of the fiber space and that of the total space, we will start to compute the mod 2 cohomology of the base space $K(\mathbb{Z}/2,2)$.

Since the base space $K(\mathbb{Z}/2,2)$ is 1-connected, by the Hurewicz theorem we have $H^1(K(\mathbb{Z}/2,2);\mathbb{Z}/2) = 0$. It follows that $E_k^{1,q} = 0$ for all $q \geq 0$ and $k \geq 2$. Consider the copy of $\mathbb{Z}/2$ at the $(0,1)$ spot on the E_2-page, it must be eliminated, but there is no non-zero group that maps to it and the only possible non-zero differential mapping out of it is d_2. Hence, we must have a copy of $\mathbb{Z}/2$ at the $(2,0)$ spot on the E_2-page to make $0 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to 0$ exact. Since this d_2 is an isomorphism, it maps generator to generator, thus we let $\iota_2 = d_2(\alpha)$ be the generator of $E_2^{2,0} = H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2) = \mathbb{Z}/2$. Notice that d_2 in this case is a transgression, and $\tau(\alpha) = d_2(\alpha) = \iota_2$.

Now consider the copy of $\mathbb{Z}/2$ at the $(0,2)$ spot on the E_2-page. This $\mathbb{Z}/2$ is $H^2(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$, hence it is generated by α^2. Again, there is no non-zero group that maps to it, but this time we have two possible non-zero differentials mapping out of it, namely d_2 and d_3. However, by the Leibniz rule, we have $d_2(\alpha^2) = d_2(\alpha) \otimes \alpha + \alpha \otimes d_2(\alpha) = 0$ (recall we are in $\mathbb{Z}/2$ coefficients). In fact, $d_2(\alpha^{2k}) = 0$ for all k, and $d_2(\alpha^{2k+1}) = d_2(\alpha) \otimes \alpha^{2k} = \iota_2 \otimes \alpha^{2k}$. Therefore, $d_2 = 0$ in this case, and to eliminate this copy of $\mathbb{Z}/2$, we must have a copy of $\mathbb{Z}/2$ at the $(3,0)$ spot on the E_2-page to make $0 \to E_2^{0,2} \xrightarrow{d_3} E_2^{3,0}$ exact at $E_2^{0,2}$. Notice that d_3 in this case is a transgression, and thus

$$d_3(\alpha^2) = \tau(\alpha^2) = \tau(Sq^1(\alpha)) = Sq^1(\tau(\alpha)) = Sq^1(\iota_2),$$

where $\alpha^2 = Sq^1(\alpha)$ because α is of dimension 1. Therefore, $E_2^{3,0}$ contains a copy of $\mathbb{Z}/2$ generated by $Sq^1(\iota_2)$. In fact, $E_2^{3,0} = \mathbb{Z}/2 = \langle Sq^1(\iota_2)\rangle$ because no other non-zero group maps to it and no non-zero differential maps out of it.

Go back to the second column ($p = 2$). We already computed $E_2^{2,0} = \mathbb{Z}/2 = \langle \iota_2 \rangle$, thus $E_2^{2,q} = \mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 = \langle \iota_2 \otimes \alpha^q \rangle$ for all $q \geq 1$. We noted above that

$d_2(\alpha^{2k+1}) = \iota_2 \otimes \alpha^{2k}$, hence the generators of all $E_2^{2,q}$ with q even are in the images of d_2, and thus $E_2^{2,q}$ with q even are all eliminated by d_2. This implies $E_2^{2,2k}$ do not persist to the E_3-page and are eliminated by $d_2\colon E_2^{0,2k+1} \to E_2^{2,2k}$. On the other hand, for $E_2^{2,2k+1} = \mathbb{Z}/2 = \langle \iota_2 \otimes \alpha^{2k+1} \rangle$, we have $d_2(\iota_2 \otimes \alpha^{2k+1}) = \iota_2 \otimes (\iota_2 \otimes \alpha^{2k}) = \iota_2^2 \otimes \alpha^{2k}$. In particular, this implies $E_2^{4,0}$ contains a copy of $\mathbb{Z}/2$ generated by ι_2^2. Note that since ι_2 is of dimension 2, we have $\iota_2^2 = Sq^2(\iota_2)$. In general, $E_2^{2,2k+1}$ do not persist to the E_3-page and are eliminated by $d_2\colon E_2^{2,2k+1} \to E_2^{4,2k}$.

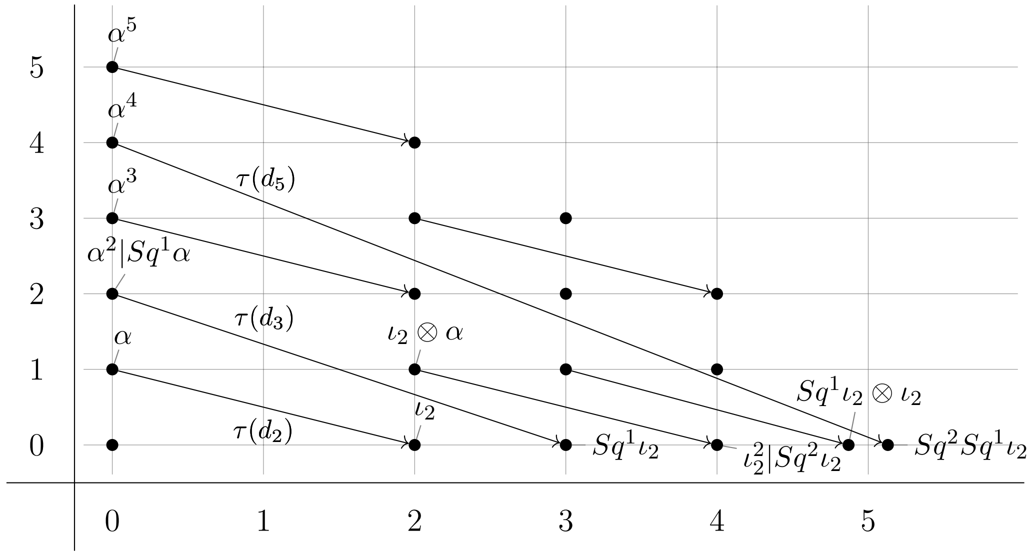
Now consider $E_2^{0,4} = \mathbb{Z}/2 = \langle \alpha^4 \rangle$. Since $\alpha^4 = (\alpha^2)^2 = Sq^2Sq^1(\alpha)$, it is transgressive and thus $E_2^{5,0}$ contains a copy of $\mathbb{Z}/2$ that is generated by $\tau(Sq^2Sq^1(\alpha)) = Sq^2Sq^1(\tau(\alpha)) = Sq^2Sq^1(\iota_2)$. Note that the transgression here is $\tau = d_5$.

One may pursue this calculation further, but we will stop at this point since the pattern of this calculation has been made clear. We end this computation with two remarks.

Remark 6.2.6. $E_2^{5,0}$ not only contains the copy of $\mathbb{Z}/2$ obtained from transgression, it also contains another copy of $\mathbb{Z}/2$ from $E_2^{3,1}$. Indeed, no non-zero group maps to $E_2^{3,1}$ (note that $E_2^{0,3}$ cannot map to it through d_3, since it does not persist to the E_3-page), and the only possible non-zero differential mapping out of $E_2^{3,1}$ is $d_2\colon E_2^{3,1} \to E_2^{5,0}$. Using the Leibniz rule, we see this copy of $\mathbb{Z}/2$ is generated by $Sq^1(\iota_2) \otimes \iota_2$.

Remark 6.2.7. It is easily verified that α^n is transgressive if and only if n is a power of 2.

The above computation is illustrated in the spectral sequence figure below.



In fact, using the result from the next section, we can show that $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$ is the polynomial ring over $\mathbb{Z}/2$ with generators $\left\{Sq^I(\iota_2)\right\}$, where I traverses over all admissible sequences of excess less than 2. This result of Serre is proved in the next section, and our computation above should at least make it plausible.

6.3 Serre's Theorem

In this section, we will prove a convenient and powerful result that gives us the cohomology rings over $\mathbb{Z}/2$ of all Eilenberg-MacLane spaces $K(\mathbb{Z}/2, q)$ for q a positive integer.

Definition 6.3.1. Let R be a graded commutative ring over a field k. A *simple system of generators* of R is an ordered set $\{x_1, x_2, \ldots\}$ such that $x_i \in R$ and the monomials $\{x_{i_1}x_{i_2}\dots x_{i_r} \mid i_1 < \cdots < i_r\}$ form a k-basis for R. Moreover, for each n, only finitely many x_i have gradation n.

Example 6.3.2. The polynomial ring $k[x]$ on one indeterminate has a simple system of generators $\left\{x, x^2, x^4, \dots, x^{2^k}, \dots\right\}$. In general, the polynomial ring $k[x_1, x_2, \dots]$ has a simple system of generators $\left\{x_i^{2^k}\right\}_{\substack{i\geq 1\\ k\geq 0}}$.

Example 6.3.3. If, over a field k, $\{x_1, x_2, \dots\}$ is a simple system of generators for a graded ring R, and $\{y_1, y_2, \dots\}$ is a simple system of generators for another graded ring S, then $\{x_1, x_2, \dots\} \cup \{y_1, y_2, \dots\}$ is a simple system of generators for $R \otimes S$.

The following theorem is due to Armand Borel, it enables us to fully compute $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$, finishing the low-dimensional computations we made in the previous section.

Theorem 6.3.4 (A. Borel)**.** *Let $F \to E \to B$ be a fibration with E acyclic, and suppose $H^*(F; \mathbb{Z}/2)$ has a simple system of transgressive generators $\{x_\alpha\}_\alpha$, then $H^*(B; \mathbb{Z}/2)$ is the polynomial ring over $\mathbb{Z}/2$ with generators $\{\tau(x_\alpha)\}_\alpha$.*

Proof. The proof uses the Serre spectral sequence and the spectral-sequence comparison theorem. We omit the proof and refer to [MT68, Chapter 9, Appendix]. □

Going back to the computation of $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$, we noted in Remark 6.2.3 that α^n is transgressive if and only if n is a power of 2, and Example 6.3.2 tells us that $\left\{\alpha^{2^k}\right\}_{k\geq 0}$ is exactly a simple system of generators for $H^*(\Omega K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$. Therefore, by Borel's theorem, $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) = \mathbb{Z}/2\left[\left\{\tau\left(\alpha^{2^k}\right)\right\}_{k\geq 0}\right]$.

However, of course, to apply Borel's theorem we must first know which elements form a simple system of transgressive generators for $H^*(F;\mathbb{Z}/2)$, which can be a quite challenging task in its own right. Fortunately, we have the following more general result.

Theorem 6.3.5 (Serre)**.** *$H^*(K(\mathbb{Z}/2,q);\mathbb{Z}/2)$ is the polynomial ring over $\mathbb{Z}/2$ with generators $\left\{Sq^I(\iota_q)\right\}$, where I runs through all admissible sequences of excess less than q.*

Proof. We make two observations.

Observation 1: if I is admissible with $e(I) > n$, then $Sq^I(\iota_n) = 0$. Indeed, let $I = \{i_1, i_2, \ldots, i_r\}$, then $e(I) > n$ implies $i_1 - i_2 - \cdots - i_r > n$, which then implies $i_1 > i_2 + \cdots + i_r + n$. Note that $Sq^I(\iota_n) = Sq^{i_1}\left(Sq^{i_2} \ldots Sq^{i_r}(\iota_n)\right)$, but the cohomology class $Sq^{i_2} \ldots Sq^{i_r}(\iota_n)$ has degree $i_2 + \cdots + i_r + n$, thus by property 2 of Theorem 3.2.1 we must have $Sq^I(\iota_n) = 0$.

Observation 2: if I is admissible with $e(I) = n$, then $Sq^I(\iota_n) = \left(Sq^J(\iota_n)\right)^{2^k}$ for some $k \geq 1$ and some admissible J with $e(J) < n$. To see this, again let $I = \{i_1, i_2, \ldots, i_r\}$, then $e(I) = n$ implies $i_1 = i_2 + \cdots + i_r + n$. Note that $Sq^I(\iota_n) = Sq^{i_1}\left(Sq^{i_2} \ldots Sq^{i_r}(\iota_n)\right)$, but the cohomology class $Sq^{i_2} \ldots Sq^{i_r}(\iota_n)$ has degree $i_2 + \cdots + i_r + n = i_1$, thus by property 3 of Theorem 3.2.1 we have $Sq^I(\iota_n) = \left(Sq^{i_2} \ldots Sq^{i_r}(\iota_n)\right)^2$. By the definition of excess, we must have $e\left(\{i_2, \ldots, i_r\}\right) \leq n$. If this excess is smaller than n, we are done. Otherwise, we repeat the process.

The proof then proceeds by induction on n. We already know the base case is true, i.e. we know $H^*(K(\mathbb{Z}/2,1);\mathbb{Z}/2) \cong \mathbb{Z}/2[\iota_1]$. Assume $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2) \cong \mathbb{Z}/2\left[\left\{Sq^I(\iota_n) \mid I \text{ admissible, } e(I) < n\right\}\right]$. Consider the loop-path fibration

$$\begin{array}{ccc} \Omega K(\mathbb{Z}/2,n+1) \cong K(\mathbb{Z}/2,n) & \longrightarrow & * \\ & & \downarrow \\ & & K(\mathbb{Z}/2,n+1) \end{array}$$

where $*$ indicates a contractible space. From Example 6.3.2, we know that $H^*(F;\mathbb{Z}/2)$ has a simple system of generators $\left\{\left(Sq^I(\iota_n)\right)^{2^k} \mid k \geq 0, I \text{ admissible, } e(I) < n\right\}$. But from the above observations, this set is equal to $\left\{Sq^J(\iota_n) \mid J \text{ admissible, } e(J) \leq n\right\}$. However, by Proposition 6.2.4, ι_n transgressive implies $Sq^J(\iota_n)$ transgressive, thus the above set is in fact a simple system of transgressive generators. By Borel's theorem, $H^*(B;\mathbb{Z}/2) = H^*(K(\mathbb{Z}/2,n+1);\mathbb{Z}/2)$ is the polynomial ring over $\mathbb{Z}/2$ with generators $\left\{\tau\left(Sq^J(\iota_n)\right)\right\} = \left\{Sq^J\left(\tau(\iota_n)\right)\right\}$
$= \left\{Sq^J(\iota_{n+1})\right\}$, where J runs through all admissible sequences with $e(J) \leq n < n+1$. This finishes the inductive step. □

Using the same method as we did in showing Proposition 6.2.5, we can show the following.

Proposition 6.3.6. *$H^*(K(\mathbb{Z},2);\mathbb{Z}/2)$ is the polynomial ring over $\mathbb{Z}/2$ generated by a two-dimensional cohomology class ι_2.*

Proof. Similar to Proposition 6.2.5. □

Similarly, using the same method as we did in proving Theorem 6.3.5 (Serre's theorem), we can prove the following.

Theorem 6.3.7. *$H^*(K(\mathbb{Z},q);\mathbb{Z}/2)$ is the polynomial ring with generators $\left\{Sq^I(\iota_q)\right\}$, where I runs through admissible sequences of excess $e(I) < q$, and the last entry in I is different from 1.*

Proof. Similar to Theorem 6.3.5. □

Remark 6.3.8. Note that in Theorem 6.3.7, there is an extra condition that the last entry in I must be different from 1. This condition exists because in this case $Sq^1(\iota_q) = 0$. To see this , we can consider the loop-path fibration

$$\begin{array}{ccc} K(\mathbb{Z}, q-1) & \longrightarrow & * \\ & & \downarrow \\ & & K(\mathbb{Z}, q) \end{array}$$

and its Serre spectral sequence. It is clear that if $Sq^1(\iota_q) \neq 0$, then $Sq^1(\iota_q)$ will persist to the E_∞-page, since no differential will eliminate it. This cannot happen in a loop-path fibration, thus we must have $Sq^1(\iota_q) = 0$.

Chapter 7

Applications: Computing Homotopy Groups of Spheres

In this final chapter, we apply all the results we have developed so far to compute the 2-components of the first five stable stems of homotopy groups. We first show that the stable homotopy groups are finite, so that we only need to compute the torsion part and we can compute it one prime at a time, then we compute the 2-components of π_1^S to π_5^S through a series of approximations to S^n.

7.1 Serre Classes

The definition of a Serre class varies from source to source, and the definition we give below is from [MT68].

Definition 7.1.1. A *Serre class* is a collection $\mathscr{C}$ of abelian groups satisfying the following axioms:

1. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, A is in $\mathscr{C}$ if and only if both A' and A'' are in $\mathscr{C}$.

2A. if $A, B \in \mathscr{C}$, then $A \otimes B \in \mathscr{C}$ and $\mathrm{Tor}(A, B) \in \mathscr{C}$.

2B. If $A \in \mathscr{C}$, then $A \otimes B \in \mathscr{C}$ for every abelian group B.

3. If $A \in \mathscr{C}$, then $H_n(K(A, 1); \mathbb{Z}) \in \mathscr{C}$ for every $n > 0$.

Remark 7.1.2. Axiom 1 implies $\mathscr{C}$ is closed under taking subgroups, quotient groups, and group extensions. Also, Axiom 2B implies Axiom 2A.

Some important examples of Serre classes include $\mathscr{C}_0$, the trivial class containing only the trivial group; $\mathscr{C}_{FG}$, the class of finitely generated abelian groups; and $\mathscr{C}_p$ for p a prime, the class of abelian torsion groups of finite exponent where the order of every element is relatively prime to p. Note that $\mathscr{C}_{FG}$ satisfies Axioms 1, 2A, and 3, and $\mathscr{C}_p$ satisfies Axioms 1, 2B, and 3.

Definition 7.1.3. A homomorphism $f \colon A \to B$ is said to be a $\mathscr{C}$-*monomorphism* if $\ker f \in \mathscr{C}$, a $\mathscr{C}$-*epimorphism* if coker $f \in \mathscr{C}$, and a $\mathscr{C}$-*isomorphism* if both $\ker f$ and coker f are in $\mathscr{C}$. (Note that a $\mathscr{C}$-isomorphism may not have an inverse.)

We define a relation $\sim$ such that $A \sim B$ if and only if there is a $\mathscr{C}$-isomorphism of A to B. Note that this relation is not necessarily symmetric (following from the comment at the end of the definition above).

Definition 7.1.4. We say A and B are *$\mathscr{C}$-isomorphic* if they are equivalent by the smallest reflexive, symmetric, and transitive relation containing the above relation. That is, A and B are $\mathscr{C}$-isomorphic if and only if there is a finite sequence $\{A = A_0, A_1, A_2, \dots, A_k = B\}$ and, for each $0 \leq i < k$, a $\mathscr{C}$-isomorphism between A_i and A_{i+1} in either direction.

Many topological theorems have mod-$\mathscr{C}$ versions, and we state the Hurewicz theorem mod $\mathscr{C}$ below, the proof of which can be found in [MT68, Chapter 11]. Note that if we take $\mathscr{C} = \mathscr{C}_0$, then the mod-$\mathscr{C}$ theorems reduce to the usual classical versions.

Theorem 7.1.5 (Hurewicz Theorem mod $\mathscr{C}$)**.** *Let X be a simply connected space, and $\mathscr{C}$ a Serre class satisfying Axioms 1, 2A, and 3. If $\pi_i(X) \in \mathscr{C}$ for all $i < n$, then $H_i(X) \in \mathscr{C}$ for all $i < n$, and the Hurewicz homomorphism $h\colon \pi_n(X) \to H_n(X)$ is a $\mathscr{C}$-isomorphism.*

Corollary 7.1.6. *Taking $\mathscr{C}$ to be $\mathscr{C}_{FG}$, the Hurewicz theorem mod $\mathscr{C}$ implies $\pi_i(S^n)$ are finitely generated for all $i > 0$ and $n > 1$.*

However, in most of our computations, we will not use the Hurewicz theorem mod $\mathscr{C}$ directly. Instead, we will use the following consequence of it. Again, the proof can be found in [MT68, Chapter 11].

Theorem 7.1.7. *Let $f : X \to Y$ be a map between 2-connected spaces such that $f^*\colon H^i(Y;\mathbb{Z}/p) \to H^i(X;\mathbb{Z}/p)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$, then $\pi_i(X)$ and $\pi_i(Y)$ have isomorphic p-components for $i < n$.*

7.2 Stable Stems π_1^S, π_2^S, π_3^S, π_4^S, and π_5^S

Computing homotopy groups is much more difficult than computing (co)homology groups because of the absense of the excision property. However, excision does not fail everywhere for homotopy groups. In fact, depending on connectivities of the spaces, there is a range of dimensions in which excision holds for homotopy groups. The exact statement is the following.

Theorem 7.2.1. *Let X be a CW-complex decomposed as the union of subcomplexes A and B with non-empty connected intersection $C = A \cup B$. If (A, C) is m-connected and (B, C) is n-connected ($m, n \geq 0$), then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is isomorphic for $i < m + n$ and epimorphic for $i = m + n$.*

As a consequence, we have the following important result. The proofs of both the above theorem and the next theorem can be found in [Hat02, Chapter 4].

Theorem 7.2.2 (Freudenthal Suspension Theorem)**.** *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is isomorphic for $i < 2n-1$ and epimorphic for $i = 2n-1$. More generally, if X is a $(n-1)$-connected CW-complex, then the same holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$.*

Observe that if we take $i = n + k$, then Theorem 7.2.2 implies $\pi_{n+k}(S^n)$ is independent of n for $n + k < 2n - 1$, that is, for $n \geq k + 2$. Therefore, for some given $k > 0$, the homotopy groups $\pi_{n+k}(S^n)$ eventually stabilizes when n becomes large enough. We call this stabilized homotopy group the k^{th} *stable stem*, and we denote it by π_k^S. The first important result we will prove in this section is the following.

Theorem 7.2.3. *The stable stems π_k^S are finite for all $k > 0$.*

Proof. Consider a positive odd integer n. Take the map $f\colon S^n \to K(\mathbb{Z}, n)$ such that the induced map $f_*\colon \pi_n(S^n) \to \pi_n(K(\mathbb{Z}, n))$ is an isomorphism. We can convert f to a fibration $X \to S^n \to K(\mathbb{Z}, n)$. Note that strictly speaking, the total space should be a different space homotopy equivalent to S^n, but we will pretend it is S^n since the results are the same up to homotopy equivalence. Consider the long exact sequence in homotopy associated with this fibration:

$$\cdots \longrightarrow \pi_{i+1}(K(\mathbb{Z}, n)) \longrightarrow \pi_i(X) \longrightarrow \pi_i(S^n) \longrightarrow \pi_i(K(\mathbb{Z}, n)) \longrightarrow \ldots.$$

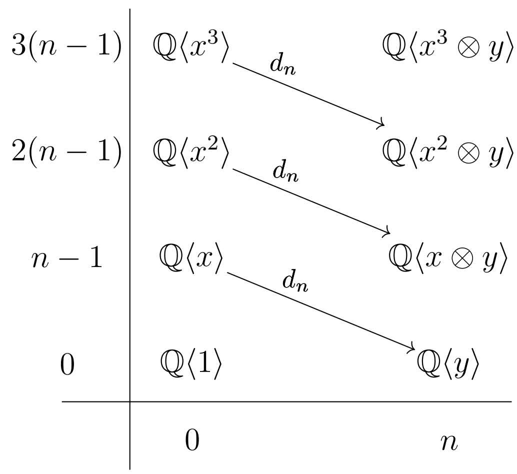
Note that $\pi_i(K(\mathbb{Z}, n)) = 0$ unless $i = n$, thus $\pi_i(X) \cong \pi_i(S^n)$ for $i > n$, and $\pi_i(X) \cong \pi_i(S^n) = 0$ for $i < n - 1$. However, $\pi_n(S^n) \to \pi_n(K(\mathbb{Z}, n))$ is an isomorphism, hence it follows that $\pi_n(X) = \pi_{n-1}(X) = 0$. Therefore,

$$\pi_i(X) \cong \begin{cases} \pi_i(S^n) & i > n; \\ 0 & i \leq n. \end{cases}$$

Now extending back along the Puppe sequence, we obtain another fibration: $\Omega K(\mathbb{Z}, n) \cong K(\mathbb{Z}, n-1) \to X \to S^n$. We consider the Serre spectral sequence of this fibration in $\mathbb{Q}$ coefficients. Since $\mathbb{Q}$ is a field, we have

$$E_2^{p,q} = H^p(S^n; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) \Longrightarrow H^{p+q}(X; \mathbb{Q}).$$

Since we assumed n is odd, we have $H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) \cong \mathbb{Q}[x]$ with $|x| = n - 1$. Therefore, we have the following Serre spectral sequence.



Note that by our computation above, X is n-connected. Therefore, the group $\mathbb{Q}\langle x \rangle$ at the $(0, n-1)$ spot must be eliminated. This implies d_n must be exact at $\mathbb{Q}\langle x \rangle$, which then implies $d_n \colon \mathbb{Q}\langle x \rangle \to \mathbb{Q}\langle y \rangle$ must be a monomorphism. However, d_n is a map between fields, thus d_n is in fact an isomorphism. After renaming, we may assume $d_n(x) = y$. Note that $d_n \colon \mathbb{Q}\langle x \rangle \to \mathbb{Q}\langle y \rangle$ being an isomorphism also implies $\mathbb{Q}\langle y \rangle$ is eliminated.

Now consider $d_n \colon \mathbb{Q}\langle x^2 \rangle \to \mathbb{Q}\langle x \otimes y \rangle$. Using the Leibniz rule, we have $d_n(x^2) = d_n(x) \otimes x + x \otimes d_n(x) = 2(x \otimes y)$. Therefore, the map $d_n \colon \mathbb{Q}\langle x^2 \rangle \to \mathbb{Q}\langle x \otimes y \rangle$ is also a monomorphism, and hence an isomorphism. In general, we may show $d_n(x^m) = m(x^{m-1} \otimes y)$. Therefore, all the d_n in the spectral sequence are isomorphisms, and thus all the copies of $\mathbb{Q}$ in the spectral sequence, except for the one at the $(0,0)$ spot, are eliminated. Therefore, $\tilde{H}^*(X; \mathbb{Q}) = 0$. But $\tilde{H}^i(X; \mathbb{Q}) \cong \tilde{H}^i(X; \mathbb{Z}) \otimes \mathbb{Q}$, thus the above result implies $\tilde{H}^i(X;\mathbb{Z})$ is finite for all $i > 0$. Using the mod-$\mathscr{C}$ theorem we introduced in the previous section, we see $\pi_i(X)$ is finite for all $i > 0$. However, recall $\pi_i(X) \cong \pi_i(S^n)$ for all $i > n$, therefore $\pi_i(S^n)$ is finite for all $i > n$. Write $i = n + k$, Theorem 7.2.2 then implies π_k^S is finite for all $k > 0$. □

Corollary 7.1.6 implies that any homotopy group $\pi_i(S^n)$, $i > 0$ and $n > 1$, can be written uniquely as

$$\pi_i(S^n) = \left(\bigoplus_p \bigoplus_{i=1}^{n_p} \mathbb{Z}/p^{r_{p,i}}\right) \oplus \mathbb{Z}^s$$

for some exponents $\{r_{p,i}\}$ and s. Now, Theorem 7.2.3 implies that any stable stem π_k^S, $k > 0$, can be written uniquely as

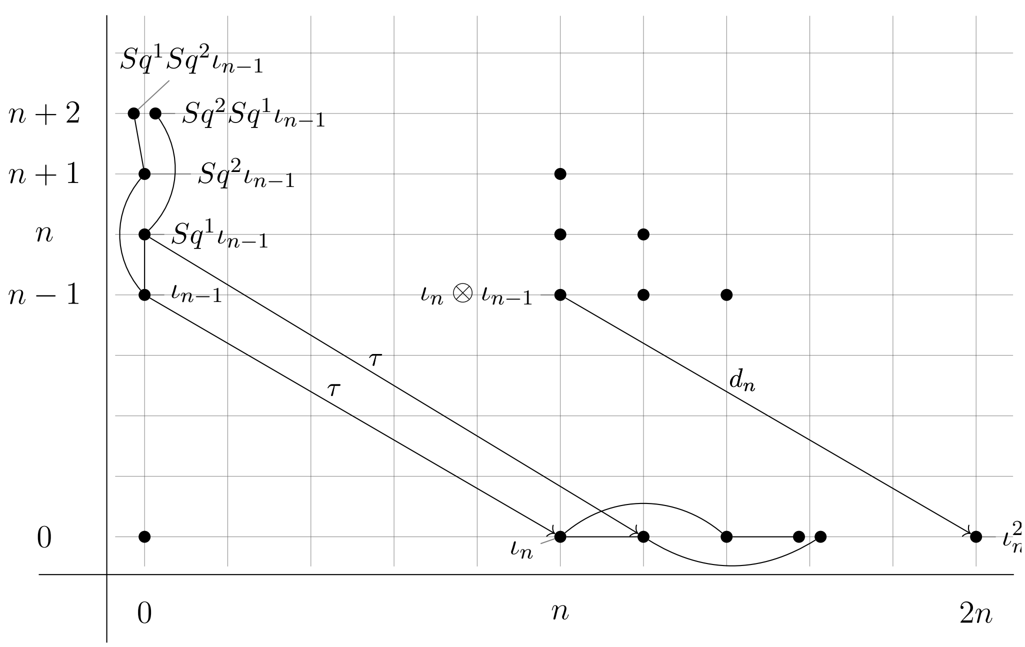
$$\pi_k^S = \bigoplus_p \bigoplus_{i=1}^{n_p} \mathbb{Z}/p^{r_{p,i}}$$

for some exponents $\{r_{p,i}\}$. The significance of this result is that we can compute the stable stems one prime at a time. In the rest of this section, we will try to compute the 2-components of π_1^S to π_5^S.

2-component of π_1^S

The strategy of our computations is the following: we start with $K(\mathbb{Z}, n)$ and regard it as our first approximation to S^n. However, this is a very coarse approximation because the mod-2 (co)homologies of $K(\mathbb{Z}, n)$ and S^n are the same only up to dimension n. Beyond dimension n, the mod-2 (co)homology of S^n becomes zero, but the mod-2 (co)homology of $K(\mathbb{Z}, n)$ is still non-zero. Therefore, we will modify $K(\mathbb{Z}, n)$ to obtain a new space X_1 that eliminates the mod-2 cohomology class in dimension $n+1$. X_1 will be a better approximation to S^n, and they will have the same 2-component in their homotopy groups in dimension $n+1$, by Theorem 7.1.7. We will then continue with this refinement to obtain better and better approximations to S^n, which will give us higher stable stems (at least their 2-components) by Theorem 7.1.7.

One caveat we want to point out before we proceed is that the computations we carry out below are only valid because we are interested in the stable stems, i.e. we allow n to be arbitrarily large. Consider the following Serre spectral sequence with $F = K(\mathbb{Z}/2, n-1)$ and $B = K(\mathbb{Z}/2, n)$. This is not the exact spectral sequence we will be using in our computations, but the moral is the same.



In the above Serre spectral sequence, consider three dimension ranges of $H^*(B;\mathbb{Z}/2)$: $(0,n)$, $[n,2n)$, and $[2n,\infty)$. In the first dimension range, everything is zero. In the second dimension range, $H^*(B;\mathbb{Z}/2)$ looks just like the Steenrod algebra $\mathcal{A}$. The structlines show the structure of $\mathcal{A}$ in this dimension range, and the only differentials that can potentially hit this range are the transgressions, therefore in this range $H^*(B;\mathbb{Z}/2)$ is “free” in some sense. Similarly, in the dimension range $[n-1,2n-2)$, $H^*(F;\mathbb{Z}/2)$ also looks like the Steenrod algebra $\mathcal{A}$, and we have drawn some of the low-dimensional structlines that mirror those on the p-axis. In this range, $H^*(F;\mathbb{Z}/2)$ is also “free” since the only differentials that can potentially leave this range are the transgressions. Now consider the third dimension range. In this range, things become quite messy, because we have all the tensor products for $p \geq n$ and $q \geq n-1$. Therefore, in this range, transgressions from the fiber space cohomology are no longer the only potential incoming differentials, and thus $H^*(B;\mathbb{Z}/2)$ stops looking like the Steenrod algebra. Instead, it will start to have the polynomial-like structure. Similar comments apply to the fiber space, beyond dimension $n-2$, $H^*(F;\mathbb{Z}/2)$ will no longer look like the Steenrod algebra.

Our computations rely on the Steenrod-algebra-like structure. That is, our descriptions of $H^*(B;\mathbb{Z}/2)$, $H^*(F;\mathbb{Z}/2)$, and the Serre spectral sequence itself, are valid only up to a neighbourhood of dimension $2n$. However, since we are interested in the stable stems, we can make n arbitrarily large so that our computations are always valid. If we were to compute homotopy groups of low-dimensional spheres that lie outside of the stable stems, then complications would arise.

With this caveat in mind, let us start computing the 2-component of π_1^S. Consider the mod-2 cohomology of $K(\mathbb{Z}, n)$. By Theorem 6.3.7, $H^*(K(\mathbb{Z}, n); \mathbb{Z}/2)$ is the polynomial ring $\mathbb{Z}/2\left[\left\{Sq^I(\iota_n)\right\}_I\right]$, where I runs through admissible sequences with $e(I) < n$, and the last entry in I is different from 1. Note that throughout our computations, we do not need to worry about the excess, since we can let n be arbitrarily large. Therefore, $H^{n+1}(K(\mathbb{Z}, n); \mathbb{Z}/2) = 0$ since $Sq^1(\iota_n) = 0$, and $H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2) = \mathbb{Z}/2 = \langle Sq^2(\iota_2)\rangle$. By the one-to-one correspondence in Theorem 2.2.1, $Sq^2(\iota_n) \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2)$ determines a homotopy class of maps in $[K(\mathbb{Z}, n), K(\mathbb{Z}/2, n+2)]$. We may denote the representative of this class by Sq^2, i.e. we now have a map

$$Sq^2 \colon K(\mathbb{Z}, n) \longrightarrow K(\mathbb{Z}/2, n+2).$$

Remark 7.2.4. This Sq^2 is of course not a cohomology operation. Instead, it is a map between two spaces such that its induced map $((Sq^2)^* : H^{n+2}(K(\mathbb{Z}/2, n+2); \mathbb{Z}/2) \to H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2))$ takes ι_{n+2} to $Sq^2(\iota_n)$, where this Sq^2 is the actual cohomology operation.

However, $K(\mathbb{Z}/2, n+2)$ fits in the loop-path fibration

$$K(\mathbb{Z}/2, n+1) \to * \to K(\mathbb{Z}/2, n+2).$$

Therefore, we have the following maps.

$$\begin{array}{ccccc} & & & & K(\mathbb{Z}/2, n+1) \\ & & & \swarrow & \\ & & * & & \\ & & \downarrow & & \\ K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) & & \end{array}$$

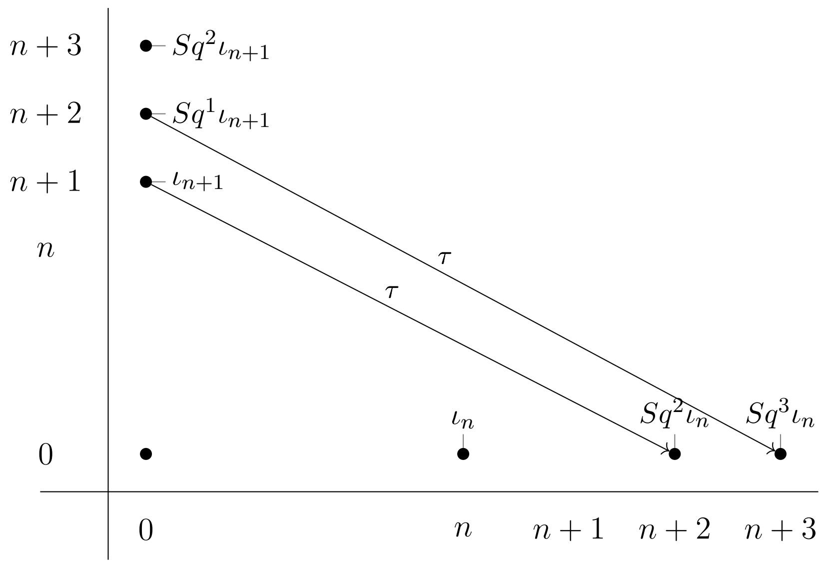
By the pullback fibration we introduced in Section 6.1, the above maps pullback to the following commutative diagram.

$$\begin{array}{ccccccc} K(\mathbb{Z}/2, n+1) & & & & & & K(\mathbb{Z}/2, n+1) \\ & \searrow & & & & \swarrow & \\ & & X_1 & \longrightarrow & * & & \\ & & \downarrow & & \downarrow & & \\ & & K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) & & \end{array}$$

Note that X_1 is the pullback space as defined in Section 6.1, and $K(\mathbb{Z}/2, n+1) \to X_1 \to K(\mathbb{Z}, n)$ forms the pullback fibration. Consider the cohomology Serre spectral sequence of this pullback fibration in $\mathbb{Z}/2$ coefficients,

$$E_2^{p,q} = H^p(K(\mathbb{Z}, n); \mathbb{Z}/2) \otimes H^q(K(\mathbb{Z}/2, n+1); \mathbb{Z}/2) \Longrightarrow H^{p+q}(X_1; \mathbb{Z}/2).$$

The spectral sequence (at least the part we are interested in right now) is illustrated below.



Note that $Sq^3(\iota_n)$ is the only generator in $H^{n+3}(B;\mathbb{Z}/2)$, because $Sq^2Sq^1(\iota_n) = Sq^2(Sq^1(\iota_n)) = Sq^2(0) = 0$.

The transgression $\tau\colon E_{n+2}^{0,n+1} \to E_{n+2}^{n+2,0}$ such that $\tau(\iota_{n+1}) = Sq^2(\iota_n)$ needs some explanation. Consider Serre's exact sequence in cohomology (Theorem 6.2.2) for the above pullback fibration diagram. The fact that the pullback fibration diagram is commutative implies the following diagram is also commutative.

$$\begin{array}{ccccccc} \ldots & \longrightarrow & H^{n+1}(K(\mathbb{Z}/2, n+1);\mathbb{Z}/2) & \xrightarrow{\tau} & H^{n+2}(K(\mathbb{Z}/2, n+2);\mathbb{Z}/2) & \longrightarrow & \ldots \\ & & \Big\downarrow \mathrm{id}^* & & \Big\downarrow (Sq^2)^* & & \\ \ldots & \longrightarrow & H^{n+1}(K(\mathbb{Z}/2, n+1);\mathbb{Z}/2) & \xrightarrow{\tau} & H^{n+2}(K(\mathbb{Z}, n);\mathbb{Z}/2) & \longrightarrow & \ldots \end{array}$$

Note that the top row corresponds to the loop-path fibration, and the bottom row corresponds to the pullback fibration.

Now consider ι_{n+1} in the top-left group $H^{n+1}(K(\mathbb{Z}/2, n+1);\mathbb{Z}/2)$. We know how the transgression maps in the loop-path fibration, thus the transgression τ in the top row maps this ι_{n+1} to ι_{n+2} in the top-right group $H^{n+2}(K(\mathbb{Z}/2, n+2); \mathbb{Z}/2)$. But by Remark 7.2.4, we have $(Sq^2)^*(\iota_{n+2}) = Sq^2(\iota_n)$. Since the diagram above is commutative, it follows that the transgression τ in the bottom row maps $\mathrm{id}^*(\iota_{n+1}) = \iota_{n+1}$ to $((Sq^2)^* \circ \tau)(\iota_{n+1}) = Sq^2(\iota_n)$. Therefore, this transgression does map as illustrated.

The other transgression $\tau\colon E_{n+3}^{0,n+2} \to E_{n+3}^{n+3,0}$ then follows easily. By Proposition 6.2.4, we have

$$\tau(Sq^1(\iota_{n+1})) = Sq^1(\tau(\iota_{n+1})) = Sq^1(Sq^2(\iota_n)) = Sq^3(\iota_n),$$

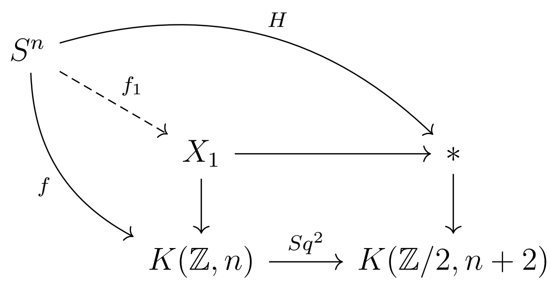
where the last equality follows from an Adem relation.

From the spectral sequence illustrated above, we observe that $H^n(X_1; \mathbb{Z}/2) = \mathbb{Z}/2$, $H^{n+1}(X_1; \mathbb{Z}/2) = 0$ because the transgression τ on the E_{n+2}-page is isomorphic (hence monomorphic), and $H^{n+2}(X_1; \mathbb{Z}/2) = 0$ because the same transgression is also epimorphic and the transgression τ on the E_{n+3}-page is monomorphic.

Now take $f\colon S^n \to K(\mathbb{Z}, n)$ such that the homotopy class $[f]$ generates $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$. Note that in cohomology, f^* is an isomorphism through dimension n and a monomorphism in dimension $n+1$. This map f fits into the pullback square as shown below.

$$\begin{array}{ccccc} & & X_1 & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ S^n & \xrightarrow{f} & K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) \end{array}$$

Note that the composition $Sq^2 \circ f$ is null-homotopic, because $K(\mathbb{Z}/2, n+2)$ is $(n+1)$-connected. Therefore, we can add a homotopy map $H\colon S^n \to *$ to the diagram above, and it will remain commutative. By the universal property of pullback, f then lifts to a map $f_1\colon S^n \to X_1$ such that the following diagram commutes.



Our computation above shows that the induced map $f_1^*\colon H^*(X_1;\mathbb{Z}/2) \to H^*(S^n;\mathbb{Z}/2)$ is an isomorphism up to (and including) dimension $n+2$. By Theorem 7.1.7, it follows that $\pi_i(X_1)$ and $\pi_i(S^n)$ have isomorphic 2-components for all $i < n+2$. Now consider the long exact sequence in homotopy associated to the pullback fibration,

$$\cdots \to \pi_{n+2}(K(\mathbb{Z}, n)) \to \pi_{n+1}(K(\mathbb{Z}/2, n+1)) \to \pi_{n+1}(X_1) \to \pi_{n+1}(K(\mathbb{Z}, n)) \to \ldots.$$

The two groups on the ends are of course 0, thus $\pi_{n+1}(K(\mathbb{Z}/2, n+1)) \to \pi_{n+1}(X_1)$ is an isomorphism. But we know $\pi_{n+1}(K(\mathbb{Z}/2, n+1)) = \mathbb{Z}/2$, thus $\pi_{n+1}(X_1) = \mathbb{Z}/2$. It then follows that the 2-component of $\pi_{n+1}(S^n)$ is also $\mathbb{Z}/2$ (n sufficiently large, in this case $n \geq 3$). That is, the 2-component of π_1^S is $\mathbb{Z}/2$.

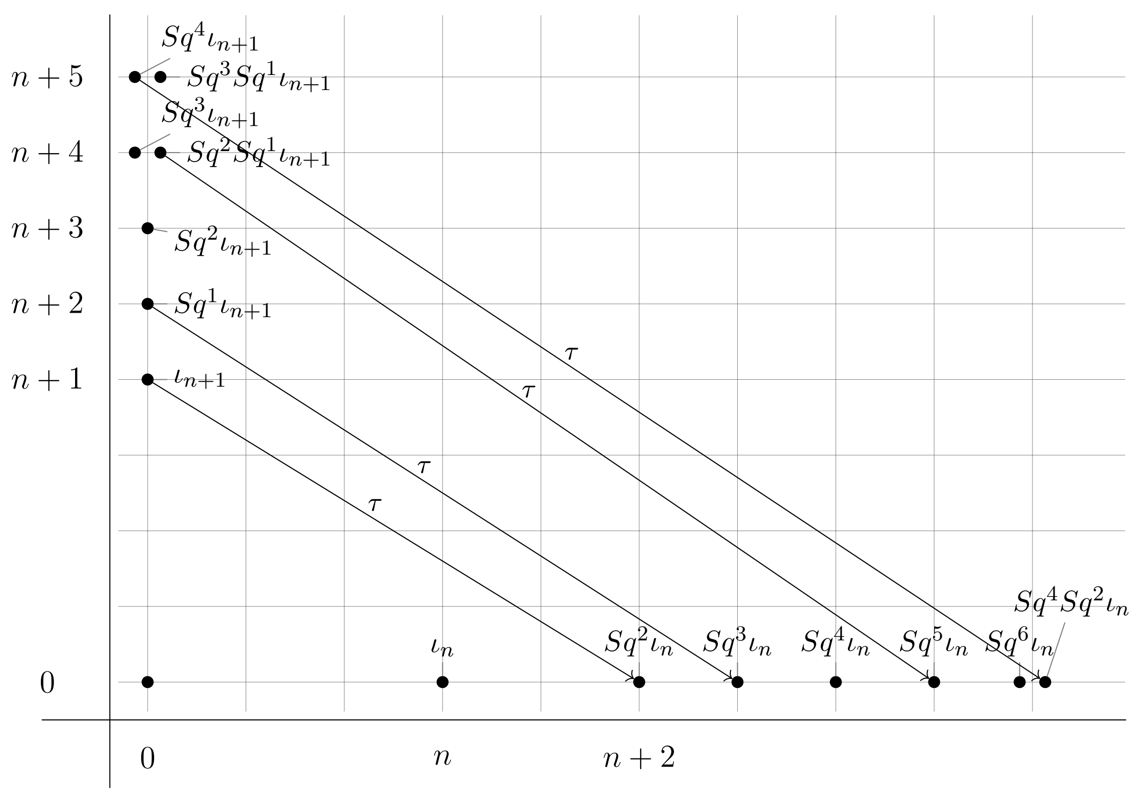
2-component of π_2^S

In the above computation, we have built the first layer of our refinement tower.

$$\begin{array}{ccccc} F_1 = K(\mathbb{Z}/2, n+1) & \xrightarrow{i_1} & X_1 & & \\ & & \big\downarrow{\scriptstyle p_1} & & \\ & & B = K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) \end{array}$$

We saw that the cohomology of X_1 agrees with the cohomology of S^n up to (and including) dimension $n+2$. This gave us the 2-component of π_1^S. What about dimensions beyond $n+2$? If the cohomologies of X_1 and S^n happened to agree in dimension $n+3$, then we would get π_2^S for free. To find out if this is the case, we must extend our above computation of the cohomology of X_1.

We need to expand the spectral sequence we illustrated above. The Adem relations play a central role in this computation, and remember that $Sq^1(\iota_n) = 0$ in the mod-2 cohomology of $B = K(\mathbb{Z}, n)$.



The transgressions (and those failed to transgress to a non-zero element) in the above illustration are computed as below.

$\tau(\iota_{n+1}) = Sq^2(\iota_n)$

$\tau(Sq^1(\iota_{n+1})) = Sq^1(\tau(\iota_{n+1})) = Sq^1(Sq^2(\iota_n)) = Sq^3(\iota_n)$

$\tau(Sq^2(\iota_{n+1})) = Sq^2(Sq^2(\iota_n)) = Sq^3(Sq^1(\iota_n)) = 0$

$\tau(Sq^3(\iota_{n+1})) = Sq^3(Sq^2(\iota_n)) = 0$

$\tau(Sq^2Sq^1(\iota_{n+1})) = Sq^2Sq^1(Sq^2(\iota_n)) = Sq^2(Sq^3(\iota_n)) = (Sq^5 + Sq^4Sq^1)(\iota_n) = Sq^5(\iota_n)$

$\tau(Sq^4(\iota_{n+1})) = Sq^4(Sq^2(\iota_n))$

$\tau(Sq^3Sq^1(\iota_{n+1})) = Sq^3Sq^1(Sq^2(\iota_n)) = Sq^3(Sq^3(\iota_n)) = Sq^5(Sq^1(\iota_n)) = 0.$

Of course, this computation can be extended indefinitely (assuming n sufficiently large, otherwise the computation would be invalid as we noted). We only present the computations up to dimension $n+5$, in higher dimensions similar computations can be carried out easily.

Observe that in the above spectral sequence figure, not all generators are eliminated. For example, $Sq^2(\iota_{n+1})$, $Sq^3(\iota_{n+1})$, and $Sq^3Sq^1(\iota_{n+1})$ are left in the kernel of τ; similarly, ι_n, $Sq^4(\iota_n)$, and $Sq^6(\iota_n)$ are left in the cokernel of τ. These generators will persist to the E_∞-page and contribute to the cohomology of X_1. Consider Serre's exact sequence in cohomology (Theorem 6.2.2) for the pullback fibration $K(\mathbb{Z}/2, n+1) \xrightarrow{i_1} X_1 \xrightarrow{p_1} K(\mathbb{Z}, n)$, which we write in a compact form below,

$$\begin{array}{ccc} H^*(K(\mathbb{Z}/2, n+1); \mathbb{Z}/2) & \xleftarrow{i_1^*} & H^*(X_1; \mathbb{Z}/2)) \\ & \searrow^{\tau} & \uparrow p_1^* \\ & & H^*(K(\mathbb{Z}, n); \mathbb{Z}/2) \end{array}$$

As we observed above, for any $x \in H^*(X_1;\mathbb{Z}/2)$, it falls into one of two kinds: 1. it comes from the kernel of τ (i.e. it comes from the elements along the vertical axis that fail to transgress and thus persist), in this case $x \in (i_1^*)^{-1}(\ker\tau)$; 2. it comes from the cokernel of τ (i.e. it comes from the elements along the horizontal axis that do not get transgressed to and thus persist), in this case $x \in p_1^*(\text{coker }\tau)$. Therefore, using the computations we carried out above, we can write down a basis for $H^*(X_1;\mathbb{Z}/2)$.

From the spectral sequence above, observe: in total degree n, we have $\iota_n \in \text{coker }\tau$, thus $H^n(X_1;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by $p_1^*(\iota_n)$; in total degrees $n+1$ and $n+2$, both kernel and cokernel are trivial, thus the mod-2 cohomology of X_1 is zero in these two dimensions; in total degree $n+3$, we have $Sq^2(\iota_{n+1}) \in \ker\tau$, thus $H^{n+3}(X_1;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by a class α such that $i_1^*(\alpha) = Sq^2(\iota_{n+1})$; in total degree $n+4$, we have $Sq^3(\iota_{n+1}) \in \ker\tau$ and $Sq^4(\iota_n) \in \text{coker }\tau$, thus $H^{n+4}(X_1;\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by a class β such that $i_1^*(\beta) = Sq^3(\iota_{n+1})$ and the class $p_1^*(Sq^4(\iota_n))$; in total degree $n+5$, we have $Sq^3Sq^1(\iota_{n+1}) \in \ker\tau$, thus $H^{n+5}(X_1;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by a class γ such that $i_1^*(\gamma) = Sq^3Sq^1(\iota_{n+1})$. This calculation can go on indefinitely.

Note that, for example, the class β is not fully determined by our above computation. Since $i_1^*(p_1^*(Sq^4(\iota_n))) = 0$, the class β may or may not contain a term $p_1^*(Sq^4(\iota_n))$ in it. We shall not be concerned about this indeterminacy, and we simply say β is the class such that $i_1^*(\beta) = Sq^3(\iota_{n+1})$.

From the above computation of the mod-2 cohomology of X_1, we obtain that $H^{n+3}(X_1;\mathbb{Z}/2) = \mathbb{Z}/2$ is non-zero, thus our second approximation to S^n, namely X_1, is only accurate up to dimension $n+2$. Therefore, to compute the 2-component of π_2^S, we must refine our approximation to eliminate the class α in dimension $n+3$.

Similar to our previous computation, by Theorem 2.2.1, $\alpha \in H^{n+3}(X_1; \mathbb{Z}/2)$ determines a homotopy class of maps in $[X_1, K(\mathbb{Z}/2, n+3)]$. We denote the representative of this class by α, and thus we have a map

$$\alpha\colon X_1 \longrightarrow K(\mathbb{Z}/2, n+3).$$

Again, by construction, the induced map $\alpha^*\colon H^{n+3}(K(\mathbb{Z}/2, n{+}3); \mathbb{Z}/2) \to H^{n+3}(X_1; \mathbb{Z}/2)$ takes ι_{n+3} to α.

However, $K(\mathbb{Z}/2, n+3)$ fits in the loop-path fibration

$$K(\mathbb{Z}/2, n+2) \to * \to K(\mathbb{Z}/2, n+3).$$

Therefore, we have the following maps.

$$\begin{array}{ccccc}
 & & & & K(\mathbb{Z}/2, n+2) \\
 & & & \swarrow & \\
 & & * & & \\
 & & \downarrow & & \\
X_1 & \xrightarrow{\alpha} & K(\mathbb{Z}/2, n+3) & &
\end{array}$$

Now by the pullback fibration in Section 6.1, the above maps pullback to the following commutative diagram.

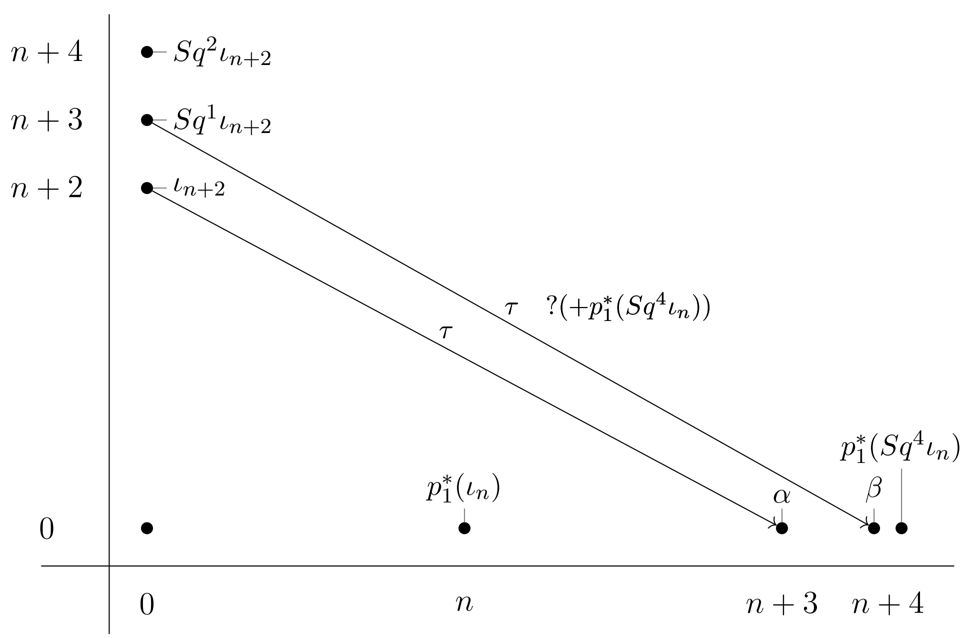
$$\begin{array}{ccccccc}
K(\mathbb{Z}/2, n+2) & & & & & & K(\mathbb{Z}/2, n+2) \\
 & \searrow & & & & \swarrow & \\
 & & X_2 & \longrightarrow & * & & \\
 & & \downarrow & & \downarrow & & \\
 & & X_1 & \xrightarrow{\alpha} & K(\mathbb{Z}/2, n+3) & &
\end{array}$$

X_2 is the pullback space, and $K(\mathbb{Z}/2, n+2) \to X_2 \to X_1$ forms a pullback fibration. As we will see, this pullback fibration will become the second layer of our refinement tower, and X_2 will become our third approximation to S^n.

Now consider the cohomology Serre spectral sequence of the above pullback fibration in $\mathbb{Z}/2$ coefficients,

$$E_2^{p,q} = H^p(X_1; \mathbb{Z}/2) \otimes H^q(K(\mathbb{Z}/2, n+2); \mathbb{Z}/2) \Longrightarrow H^{p+q}(X_2; \mathbb{Z}/2).$$

The low-dimensional part of this spectral sequence is illustrated below. The vertical axis is the mod-2 cohomology of $K(\mathbb{Z}/2, n+2)$, which we know looks just like the Steenrod algebra $\mathcal{A}$; the horizontal axis is the mod-2 cohomology of X_1, which we already computed before.



The transgression $\tau\colon E_{n+3}^{0,n+2} \to E_{n+3}^{n+3,0}$ again follows from construction. Consider Serre's exact sequence in cohomology for the above pullback fibration (and also for the loop-path fibration from which the pullback fibration is induced). The above pullback fibration diagram being commutative implies the following diagram also commutes.

$$\begin{array}{ccccccc} \ldots & \longrightarrow & H^{n+2}(K(\mathbb{Z}/2,n+2);\mathbb{Z}/2) & \xrightarrow{\tau} & H^{n+3}(K(\mathbb{Z}/2,n+3);\mathbb{Z}/2) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \mathrm{id}^*} & & \downarrow{\scriptstyle \alpha^*} & & \\ \ldots & \longrightarrow & H^{n+2}(K(\mathbb{Z}/2,n+2);\mathbb{Z}/2) & \xrightarrow{\tau} & H^{n+3}(X_1;\mathbb{Z}/2) & \longrightarrow & \ldots \end{array}$$

The top row corresponds to the loop-path fibration, and the bottom row corresponds to the pullback fibration. Consider $\iota_{n+2} \in H^{n+2}(K(\mathbb{Z}/2, n+2); \mathbb{Z}/2)$, the commutativity of the above diagram implies

$$\tau(\iota_{n+2}) = \tau(\mathrm{id}^*(\iota_{n+2})) = \alpha^*(\tau(\iota_{n+2})) = \alpha^*(\iota_{n+3}) = \alpha,$$

where the different transgressions should raise no confusion, even though they share the same symbol τ; also, the last equality follows from the comment we made immediately after defining the map α. Therefore, this transgression $\tau\colon E_{n+3}^{0,n+2} \to E_{n+3}^{n+3,0}$ indeed maps ι_{n+2} to α.

Now consider the transgression $\tau\colon E_{n+4}^{0,n+3} \to E_{n+4}^{n+4,0}$. This transgression calculates

$$\tau(Sq^1(\iota_{n+2})) = Sq^1(\tau(\iota_{n+2})) = Sq^1(\alpha).$$

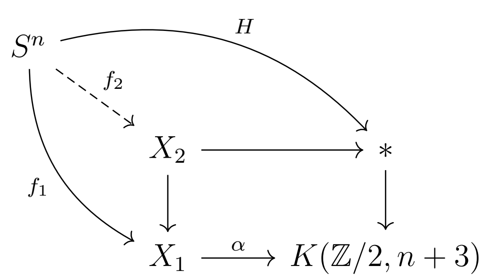
But what is $Sq^1(\alpha)$? Recall that $i_1^*(\alpha) = Sq^2(\iota_{n+1})$, thus by naturality of the squares, $i_1^*(Sq^1(\alpha)) = Sq^1(i_1^*(\alpha)) = Sq^1(Sq^2(\iota_{n+1})) = Sq^3(\iota_{n+1})$. But recall we also have $i_1^*(\beta) = Sq^3(\iota_{n+1})$ and $i_1^*(p_1^*(Sq^4(\iota_n))) = 0$, therefore

$$\tau(Sq^1(\iota_{n+2})) = Sq^1(\alpha) = \beta + s \cdot p_1^*(Sq^4(\iota_n)),$$

where s is either 0 or 1. Hence, we have an indeterminacy in the transgression $\tau\colon E_{n+4}^{0,n+3} \to E_{n+4}^{n+4,0}$, which is indicated by the annotation $?(+p_1^*(Sq^4\iota_n))$ next to it in the above spectral sequence figure. We will see later that this indeterminacy poses no issue. This computation can of course continue indefinitely, provided we have computed the mod-2 cohomology of X_1 indefinitely. However, we will postpone this computation to the next section, since we are only interested in the 2-component of π_2^S here.

From the spectral sequence illustrated above, we observe that $H^n(X_2;\mathbb{Z}/2) = \mathbb{Z}/2$, $H^{n+1}(X_2;\mathbb{Z}/2) = 0$ because there is nothing in dimension $n+1$, $H^{n+2}(X_2;\mathbb{Z}/2) = 0$ because the transgression τ on the E_{n+3}-page is isomorphic (hence monomorphic), and $H^{n+3}(X_2;\mathbb{Z}/2) = 0$ because the same transgression is also epimorphic and the transgression τ on the E_{n+4}-page is monomorphic.

Moreover, by the same argument as before, the composition $\alpha\circ f_1\colon S^n \to K(\mathbb{Z}/2, n+3)$ is null-homotopic, because $K(\mathbb{Z}/2, n+3)$ is $(n+2)$-connected. Therefore, by the universal property of pullback, $f_1\colon S^n \to X_1$ can be lifted to a map $f_2\colon S^n \to X_2$ such that the following diagram commutes.



Our computation shows that the induced map $f_2^*\colon H^*(X_2;\mathbb{Z}/2) \to H^*(S^n;\mathbb{Z}/2)$ is an isomorphism up to (and including) dimension $n+3$. By Theorem 7.1.7, we then have that $\pi_i(X_2)$ and $\pi_i(S^n)$ have isomorphic 2-components for all $i < n+3$. Now consider the long exact sequence in homotopy associated to the pullback fibration,

$$\cdots \to \pi_{n+3}(X_1) \to \pi_{n+2}(K(\mathbb{Z}/2, n+2)) \to \pi_{n+2}(X_2) \to \pi_{n+2}(X_1) \to \dots,$$

we claim $\pi_{n+3}(X_1) = \pi_{n+2}(X_1) = 0$. Indeed, from the long exact sequence at the end of the previous section (the long exact sequence in homotopy associated to the pullback fibration from our first refinement), we have $\pi_{n+2}(X_1) \cong \pi_{n+2}(K(\mathbb{Z}/2, n+1))$ and $\pi_{n+3}(X_1) \cong \pi_{n+3}(K(\mathbb{Z}/2, n+1))$, but the two homotopy groups on the right hand side are clearly zero, thus $\pi_{n+3}(X_1) = \pi_{n+2}(X_1) = 0$. Therefore, we have $\pi_{n+2}(X_2) \cong \pi_{n+2}(K(\mathbb{Z}/2, n+2)) \cong \mathbb{Z}/2$. It follows that the 2-component of $\pi_{n+2}(S^n)$ is also $\mathbb{Z}/2$ (again, assuming n sufficiently large, in this case $n \geq 4$). Hence, the 2-component of π_2^S is $\mathbb{Z}/2$.

2-components of π_3^S, π_4^S, and π_5^S

The above computation shows that X_2 is indeed our third approximation to S^n. We now have completed the second layer of our refinement tower, it stacks to the first layer in the following way.

$$\begin{array}{ccccc} F_2 = K(\mathbb{Z}/2, n+2) & \xrightarrow{i_2} & X_2 & & \\ & & \downarrow{\scriptstyle p_2} & & \\ F_1 = K(\mathbb{Z}/2, n+1) & \xrightarrow{i_1} & X_1 & \xrightarrow{\alpha} & K(\mathbb{Z}/2, n+3) \\ & & \downarrow{\scriptstyle p_1} & & \\ & & K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) \end{array}$$

Now we face a similar question: how good is this approximation X_2? In other words, does the isomorphism between $H^i(X_2;\mathbb{Z}/2)$ and $H^i(S^n;\mathbb{Z}/2)$ extend beyond dimension $n+3$? To answer this question, consider the Serre spectral sequence $E_2^{p,q} = H^p(X_1; H^q(K(\mathbb{Z}/2, n+2);\mathbb{Z}/2)) \Rightarrow H^{p+q}(X_2;\mathbb{Z}/2)$ we illustrated in the previous section. We computed, in this spectral sequence, $\tau(Sq^1(\iota_{n+2})) = \beta + s\cdot p_1^*(Sq^4(\iota_n))$, where s is either 0 or 1. Observe that

$$\text{coker } \tau = \frac{H^{n+4}(X_1;\mathbb{Z}/2)}{\text{im } \tau} = \frac{\mathbb{Z}/2\,\langle\beta\rangle \oplus \mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle}{\text{im } \tau}.$$

If $s = 0$, then clearly

$$\text{coker } \tau = \frac{\mathbb{Z}/2\,\langle\beta\rangle \oplus \mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle}{\mathbb{Z}/2\,\langle\beta\rangle} \cong \mathbb{Z}/2\,\left\langle p_1^*(Sq^4(\iota_n))\right\rangle.$$

On the other hand, if $s = 1$, then we have

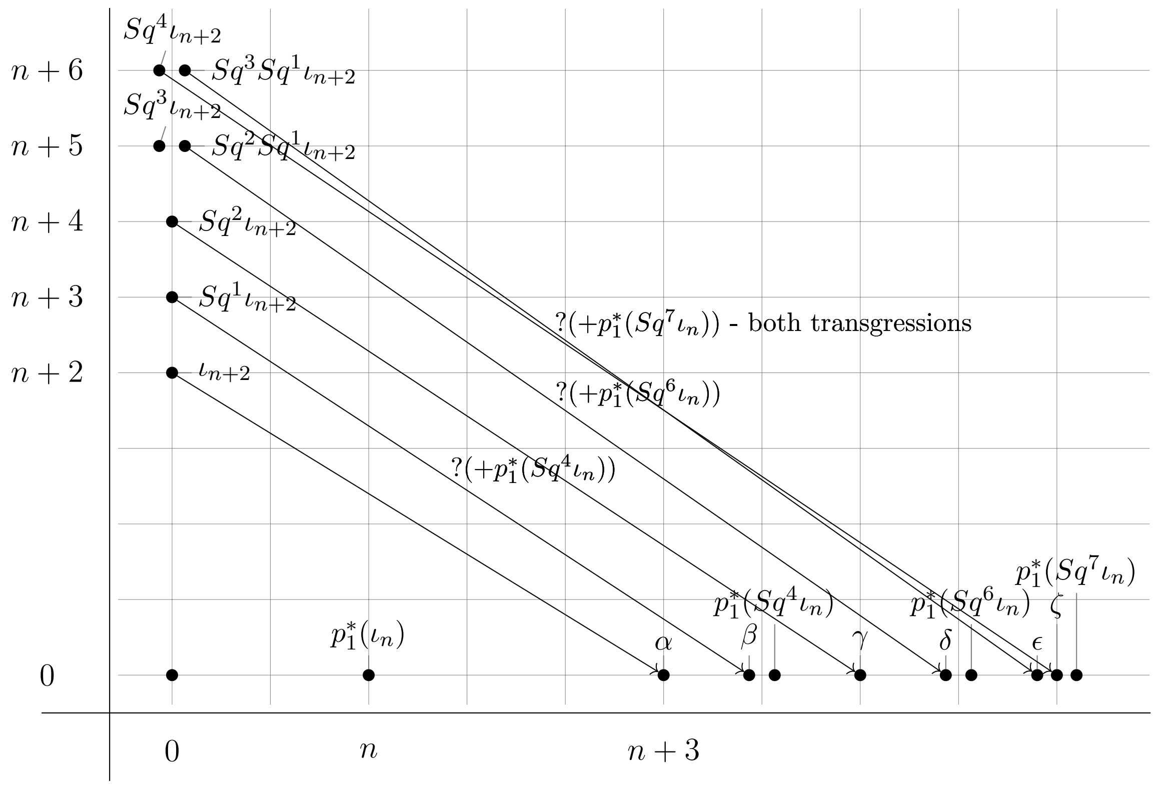
$$\text{coker } \tau = \frac{\mathbb{Z}/2\,\langle\beta\rangle \oplus \mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle}{\mathbb{Z}/2\,\langle\beta + p_1^*(Sq^4(\iota_n))\rangle},$$

and this quotient group can be viewed as either $\mathbb{Z}/2\,\langle\beta\rangle$ or $\mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle$. Hence, coker $\tau \cong \mathbb{Z}/2$, implying $H^{n+4}(X_2;\mathbb{Z}/2)$ contains at least one copy of $\mathbb{Z}/2$. Therefore, our third approximation X_2 is only good through dimension $n+3$. However, since we mod out by the subgroup generated by $\beta + p_1^*(Sq^4(\iota_n))$, we can regard β and $p_1^*(Sq^4(\iota_n))$ as the same. Therefore, we simply have coker $\tau \cong \mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle$.

Similar to the argument we gave when computing the mod-2 cohomology of X_1, we now consider Serre's exact sequence in cohomology for the pullback fibration $K(\mathbb{Z}/2, n+2) \xrightarrow{i_2} X_2 \xrightarrow{p_2} X_1$, which we write compactly as

$$\begin{array}{ccc} H^*(K(\mathbb{Z}/2, n+2); \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^*(X_2; \mathbb{Z}/2) \\ & \searrow^{\tau} & \uparrow p_2^* \\ & & H^*(X_1; \mathbb{Z}/2) \end{array}$$

Any $x \in H^*(X_2; \mathbb{Z}/2)$ then falls into one of two kinds: 1. $x \in (i_2^*)^{-1}(\ker \tau)$; 2. $x \in p_2^*(\text{coker } \tau)$. It follows from the computation we just did that $H^{n+4}(X_2; \mathbb{Z}/2)$ contains at least one copy of $\mathbb{Z}/2$ generated by $p_2^*(p_1^*(Sq^4(\iota_n)))$. For the purpose of computing higher stable homotopy groups, we may wish to push this computation further. That is, we will compute more transgressions in the above spectral sequence to obtain a better description of the mod-2 cohomology of X_2. The expanded illustration of the spectral sequence for $H^*(X_2; \mathbb{Z}/2)$ is shown below.



In the previous section, we did not compute the mod-2 cohomology of X_1 to dimension $n+7$ (recall we only computed up to dimension $n+5$), but one may carry out this computation easily by themselves. We mention the following:

$$\delta \text{ is the class such that } i_1^*(\delta) = (Sq^5 + Sq^4Sq^1)(\iota_{n+1})$$

$$\epsilon \text{ is the class such that } i_1^*(\epsilon) = Sq^5Sq^1(\iota_{n+1})$$

$$\zeta \text{ is the class such that } i_1^*(\zeta) = Sq^4Sq^2(\iota_{n+1}).$$

The transgressions (and those failed to transgress to a non-zero element) in the above illustration are computed as below. Recall we already computed $\tau(\iota_{n+2})$ and $\tau(Sq^1(\iota_{n+2}))$.

$\tau(Sq^2(\iota_{n+2})) = Sq^2(\tau(\iota_{n+2})) = Sq^2(\alpha)$. Since $Sq^2(\alpha)$ lives in dimension $n+5$, it is either γ or 0. But by definition, $i_1^*(\alpha) = Sq^2(\iota_{n+1})$, thus $i_1^*(Sq^2(\alpha)) = Sq^2(i_1^*(\alpha)) = Sq^2(Sq^2(\iota_{n+1})) = Sq^3(Sq^1(\iota_{n+1})) = i_1^*(\gamma)$, where the last equality also follows from definition. Therefore, we must have $Sq^2(\alpha) = \gamma$, and thus $\tau(Sq^2(\iota_{n+2})) = \gamma$. This computation shows that the transgression τ on the E_{n+5}-page is isomorphic (hence monomorphic), thus in total degree $n+4$, $\ker\tau = 0$ and coker $\tau = \mathbb{Z}/2\,\langle p_1^*(Sq^4(\iota_n))\rangle$. Therefore, $H^{n+4}(X_2;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by $p_2^*(p_1^*(Sq^4(\iota_n)))$.

Using the same trick of acting by i_1^*, we have that

$$i_1^*(Sq^2Sq^1(\alpha)) = Sq^2Sq^1(Sq^2(\iota_{n+1})) = (Sq^5 + Sq^4Sq^1)(\iota_{n+1}) = i_1^*(\delta),$$

where the last equality follows from definition. But we also have $i_1^*(p_1^*(Sq^6(\iota_n))) = 0$. Therefore, $\tau(Sq^2Sq^1(\iota_{n+2})) = \delta + s \cdot p_1^*(Sq^6(\iota_n))$, where s is either 0 or 1.

Now to compute $\tau(Sq^3(\iota_{n+2}))$, we need some Bockstein relations. We have no intention of expanding on this topic, thus we simply give them as facts and refer to [MT68, Chapter 11, Chapter 12]. If we denote the differentials in the Bockstein spectral sequence by d_i^B, then $d_1^B(\gamma) = Sq^1(\gamma) = 0$. Of course, we do not pretend this is a trivial fact, but its proof does not concern us here. From this, we have

$$\tau(Sq^3(\iota_{n+2})) = \tau(Sq^1Sq^2(\iota_{n+2})) = Sq^1(\tau(Sq^2(\iota_{n+2}))) = Sq^1(\gamma) = 0.$$

Since we already showed that the transgression τ on the E_{n+5}-page is isomorphic (hence epimorphic), in total degree $n+5$, we have coker $\tau = 0$ and $\ker\tau = \mathbb{Z}/2\,\langle Sq^3(\iota_{n+2})\rangle$. Therefore, $H^{n+5}(X_2;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by a class A such that $i_2^*(A) = Sq^3(\iota_{n+2})$.

Also, using the same argument as we gave at the beginning of this section, we have that in total degree $n+6$, coker $\tau \cong \mathbb{Z}/2\,\langle p_1^*(Sq^6(\iota_n))\rangle$. It is then left to compute $\ker\tau$, which means we need to compute the transgression τ on the E_{n+7}-page.

Again, using the i_1^* trick, we can show

$$\tau(Sq^4(\iota_{n+2})) = \zeta + s\cdot Sq^7(\iota_n), \text{ and } \tau(Sq^3Sq^1(\iota_{n+2})) = \epsilon + t\cdot Sq^7(\iota_n),$$

where s and t are either 0 or 1. This shows in total degree $n+6$, $\ker\tau = 0$. That is, the transgression τ on the E_{n+7}-page is monomorphic. Combined with the result we obtained above, this shows that $H^{n+6}(X_2;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by $p_2^*(p_1^*(Sq^6(\iota_n)))$. Also, using the same argument, in total degree $n+7$ the cokernel of τ is $\mathbb{Z}/2$ generated by $p_1^*(Sq^7(\iota_n))$. This computation can continue indefinitely, and we will stop here.

Since X_2 is only a good approximation up to dimension $n+3$, we must refine our approximation again in order to compute stable homotopy groups π_k^S for $k \geq 3$. In particular, to compute the 2-component of π_3^S, we must eliminate the class $p_2^*(p_1^*(Sq^4(\iota_n)))$ in dimension $n+4$.

From our previous computation, the first idea may be to obtain a map $X_2 \to K(\mathbb{Z}/2, n{+}4)$, given by the class $p_2^*(p_1^*(Sq^4(\iota_n))) \in H^{n+4}(X_2;\mathbb{Z}/2)$ and Theorem 2.2.1. This construction certainly exists, which is guaranteed by Theorem 2.2.1. However, if we pursue this line of computation and form the pullback fibration $K(\mathbb{Z}/2, n+3) \to X_3 \to X_2$, we will see (after an easy calculation) that the class $Sq^1(\iota_{n+3}) \in H^{n+4}(K(\mathbb{Z}/2, n+3);\mathbb{Z}/2)$ fails to transgress. This implies in total degree $n+4$,

$\ker\tau$ is non-trivial, thus $H^{n+4}(X_3;\mathbb{Z}/2)$ is also non-trivial. Therefore, X_3 obtained in this way will not be a better approximation than X_2. We may next try a map $X_2 \to K(\mathbb{Z}/4, n+4)$, but it will fail for a similar reason.

In fact, the correct choice here is to obtain a map $X_2 \to K(\mathbb{Z}/8, n+4)$. After we induce the pullback fibration in the usual way, we obtain our third refinement, and the pullback space X_3 will be our fourth approximation to S^n. Again, this refinement layer stacks to our refinement tower as below.

$$\begin{array}{ccccc}
K(\mathbb{Z}/8,n+3) & \xrightarrow{i_3} & X_3 & & \\
& & \big\downarrow{\scriptstyle p_3} & & \\
K(\mathbb{Z}/2,n+2) & \xrightarrow{i_2} & X_2 & \xrightarrow{\text{“}Sq^4\iota_n\text{”}} & K(\mathbb{Z}/8,n+4) \\
& & \big\downarrow{\scriptstyle p_2} & & \\
K(\mathbb{Z}/2,n+1) & \xrightarrow{i_1} & X_1 & \xrightarrow{\alpha} & K(\mathbb{Z}/2,n+3) \\
& & \big\downarrow{\scriptstyle p_1} & & \\
& & K(\mathbb{Z},n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2,n+2)
\end{array}$$

Consider the Serre spectral sequence,

$$E_2^{p,q} = H^p(X_2;\mathbb{Z}/2) \otimes H^q(K(\mathbb{Z}/8,n+3);\mathbb{Z}/2) \Longrightarrow H^{p+q}(X_3;\mathbb{Z}/2).$$

Computing the transgressions in this spectral sequence as before (the pattern should be clear by now), we will see that on the E_{n+4}- through E_{n+7}-pages the transgressions are isomorphic, but on the E_{n+8}-page the transgression fails to be monomorphic. Lift $f_2\colon S^n \to X_2$ to a map $f_3\colon S^n \to X_3$ through the pullback, the computation of the transgressions then implies $f_3^*\colon H^*(X_3;\mathbb{Z}/2) \to H^*(S^n;\mathbb{Z}/2)$ is an isomorphism up to (and including) dimension $n+6$, since the transgression on the E_{n+8}-page failing to be monomorphic implies the kernel of τ is non-trivial in total degree $n + 7$, and thus $H^{n+7}(X_3; \mathbb{Z}/2)$ is non-trivial. Theorem 7.1.7 then implies $\pi_i(X_3)$ and $\pi_i(S^n)$ have isomorphic 2-components up to dimension $i = n + 5$. Therefore, our fourth approximation X_3 not only gives us the 2-component of the third stable stem π_3^S, but also the 2-components of π_4^S and π_5^S.

Consider the long exact sequence in homotopy associated to the third refinement layer (the pullback fibration $K(\mathbb{Z}/8, n+3) \to X_3 \to X_2$),

$$\cdots \to \pi_{n+k+1}(X_2) \to \pi_{n+k}(K(\mathbb{Z}/8, n+3)) \to \pi_{n+k}(X_3) \to \pi_{n+k}(X_2) \to \dots,$$

note that $\pi_i(X_2) = 0$ for all $i > n + 2$. Therefore, the above long exact sequence shows that $\pi_{n+3}(X_3) = \mathbb{Z}/8$, and $\pi_{n+4}(X_3) = \pi_{n+5}(X_3) = 0$. By Theorem 7.1.7, we then have that the 2-component of π_3^S is $\mathbb{Z}/8$, and the 2-components of π_4^S and π_5^S are both 0.

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